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Three systems of waveguide discontinuities are considered. The first system is that of multiple apertures of arbitrary shape in the transverse plane between two cylindrical waveguides. The second system consists of metallic obstacles in a rectangular waveguide that are uniform along the narrow side of waveguide but are otherwise of arbitrary shape and thickness, i.e., a system of inductive posts. The third system consists of metallic obstacles in a rectanguar waveguide that are uniform along the broad side of the waveguide, but
(cont'd)

20. Abstract (continued)

are otherwise of arbitrary cross section and thickness, i.e., a system of capacitive posts. Common between the first and second systems are the inductive windows and strips in a rectangular waveguide, and between the first and third systems are the capacitive windows and strips in a rectangular waveguide.

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Chapter 1

INTRODUCTION

An important topic in electromagnetic field analysis is the analysis of waveguide discontinuities. This is because these discontinuities are basic components in many microwave devices, so that an exact knowledge of their characteristics is essential. Also, the analysis of waveguide discontinuities, from a theoretical point of view, is an interesting problem. The theory developed has found immediate application to some types of discontinuities, and has indeed deepened the understanding of electromagnetics in general.

Despite the extensive consideration of the subject, however, the analysis has been confined to a small set of discontinuities. Specifically, discontinuities that are coincident with or have regular cross sections symmetric about one waveguide cross section and are uniform along one of the waveguide axes have been the most frequently treated ones. Furthermore, in a typical situation, only a limited number of discontinuities, usually one or two, of the same geometry would have been assumed. General systems of discontinuities, it appears, have never been considered before.

In analyzing waveguide discontinuities, the complete field solution is of very little interest. Rather, the effect of the discontinuities on the incident modes is what must be described as accurately as possible. From an engineering perspective, descriptions employing networks of lumped elements are preferred. Any such network must, for a structure that is both lossless and reciprocal, obey the two basic network laws: the conservation of complex power law and the reciprocity law.

One of the earliest methods used for determining the elements of the network representation of waveguide discontinuities has been the "Variational Method" [1]. In this method, the network elements are so expressed that they are stationary with respect to arbitrary small variations of some field related quantity about its true value. That is, the network elements would be accurate to an order one higher than that of the trial quantity. A judicious choice of the trial quantity can then lead to remarkably accurate results.

Although ingenious and powerful, the application of the Variational Method has primarily been restricted to single discontinuities of simple geometries and moderate sizes that are symmetric about a waveguide cross section. This is because finding out the appropriate trial quantity, except perhaps for some simple geometries, requires a great deal of insight into the problem, and, at times, solving another problem. Furthermore, although the results obtained can be improved in a systematic manner using standard procedure [2, Section 7-6], the process is quite laborious. An extensive collection of theoretical and numerical results for a large variety of waveguide discontinuities obtained using this method can be found in the Waveguide Handbook [3].

The Variational Method has gradually given way to the more general viewpoint of the "Method of Moments" [4]. Because of its variational character [5], and combined with the very fast development of computer systems and software techniques, the Method of Moments has become a basic tool in the study of waveguide discontinuities, as well as of many other areas of electromagnetics. Here, the network representation is determined in terms of some field related quantity, for which an operator, usually integral, equation is to be solved. The solution proceeds by expanding the unknown field quantity as a linear combination of some appropriate functions. Enforcing the governing equation in some way then leads to a matrix equation for the coefficients of the expansion, which can be solved using a matrix solution routine [6].

Moment solutions are analytically simple, in the sense that no excessive manipulations are needed in order to minimize the computational phase of the solution. Furthermore, the amount of effort expended in solving for many discontinuities is the same as that is put solving for only one. Since computer codes are always written, the automatic improvement of the solutions is very easily done. The Method of Moments, therefore, is well suited for solving systems of waveguide discontinuities, provided a suitable analysis can be given. Recently, the application of the Method of Moments to waveguide discontinuities has been an area of active research. Specifically, single and triple inductive posts of circular cross section have been considered in [7], [8], where many of their characteristics have been discussed.

As is pointed out earlier, general systems of waveguide discontinuities have yet to be considered. It is the purpose of this dissertation to consider three such systems.

The first system is that of multiple apertures of arbitrary shape in the transverse plane between two cylindrical waveguides. The second system consists of metallic obstacles in a rectangular waveguide that are uniform along the narrow side of the waveguide, but are otherwise of arbitrary shape and thickness, i.e., a system of inductive posts. The third system consists of metallic obstacles in a rectangular waveguide that are uniform along the broad side of the waveguide, but are otherwise of arbitrary shape and thickness, i.e., a system of capacitive posts. Common between the first and second systems are the inductive windows and strips in a rectangular waveguide, and between the first and third systems are the capacitive windows and strips in a rectangular waveguide.

The system of multiple apertures is considered in Chapter 2, and is depicted in Figure 1 there. Assuming a multi-mode exciting field, a modal expansion is used to express the field in the two waveguides. A field equivalence theorem and Galerkin procedure are then utilized to obtain the generalized network representation of the apertures. This representation is shown to obey the two basic network laws: the conservation of complex power law and the reciprocity law. The scattering matrix is then deduced from the generalized network representation, and its properties are examined. The analysis is subsequently specialized to the problems of inductive and capacitive windows in a rectangular waveguide, where the dominant mode is the only incident wave. The impedance matrices of the windows are also obtained, and readily realized by networks of shunt reactive elements.

The system of inductive posts, which can be seen depicted in Figure 1 of Chapter 3, is then considered. The exciting field is taken to be the dominant waveguide mode. A complete field analysis of the problem is given. The analysis is quite general, and results in an integral equation for the currents induced on the posts. Later, this equation is solved through a Galerkin procedure. The scattering and impedance matrices describing the effect of the posts on the dominant waveguide mode are then obtained and examined in detail. The impedance matrix is subsequently realized by a two-port T-network of reactive elements. The computed reactances of some post configurations are also reported.

The system of capacitive posts is then considered in Chapter 4, where it is depicted in Figure 1. Since this system is dual to that of the inductive posts, the analysis is drawn on similar lines. Specifically, a complete field analysis leads to an integral equation for the currents induced on the posts due to an incident dominant mode wave, which is later solved through a Galerkin procedure. The scattering and impedance matrix representations of the posts are then extracted from the analysis, and their properties are discussed. The impedance matrix is subsequently realized by a two-port T-network of reactive elements. The computed reactances of some selected post configurations are given.

A word about the organization of the dissertation is in order. Each system of discontinuities is considered completely independent of the others. This is believed to be the best approach to the subject, despite the repetition of some of the definitions and proofs. A discussion at the end of each chapter points out the important results of the analysis and closely related works. Final remarks are given in Chapter 5.

Chapter 2

MULTIPLE APERTURES IN THE TRANSVERSE PLANE BETWEEN TWO CYLINDRICAL WAVEGUIDES

Consider a system of apertures S^1, S^2,...,S^p of arbitrary shape located in the transverse plane between two uniform cylindrical waveguides A and B extending along the z-axis. The mediums filling waveguides A and B are assumed linear, homogeneous, isotropic, and dissipation free, and are therefore characterized by the real scalar permittivities ε_a and ε_b, and the real scalar permeabilities μ_a and μ_b, respectively. Figure 1 shows the problem at hand.

1. Basic Formulation

Let a multi-mode field be incident in waveguide A. Part of the incident field is then reflected into waveguide A, while the rest of it is transmitted into waveguide B.

The total z-transverse field in both waveguides can be expressed in modal form as [2, Section 8-2]

$$\underline{E}_t = \begin{cases} \sum\limits_i c_i\, e^{-\gamma_{ai} z}\, \underline{e}_{ai} + \sum\limits_i a_i\, e^{\gamma_{ai} z}\, \underline{e}_{ai} & z < 0 \\[2ex] \sum\limits_i b_i\, e^{-\gamma_{bi} z}\, \underline{e}_{bi} & z > 0 \end{cases}$$

$$\underline{H}_t = \begin{cases} \sum\limits_i c_i \eta_{ai}\, e^{-\gamma_{ai} z}\, \underline{z} \times \underline{e}_{ai} - \sum\limits_i a_i \eta_{ai}\, e^{\gamma_{ai} z}\, \underline{z} \times \underline{e}_{ai} & z < 0 \\[2ex] \sum\limits_i b_i \eta_{bi}\, e^{-\gamma_{bi} z}\, \underline{z} \times \underline{e}_{bi} & z > 0. \end{cases} \tag{1}$$

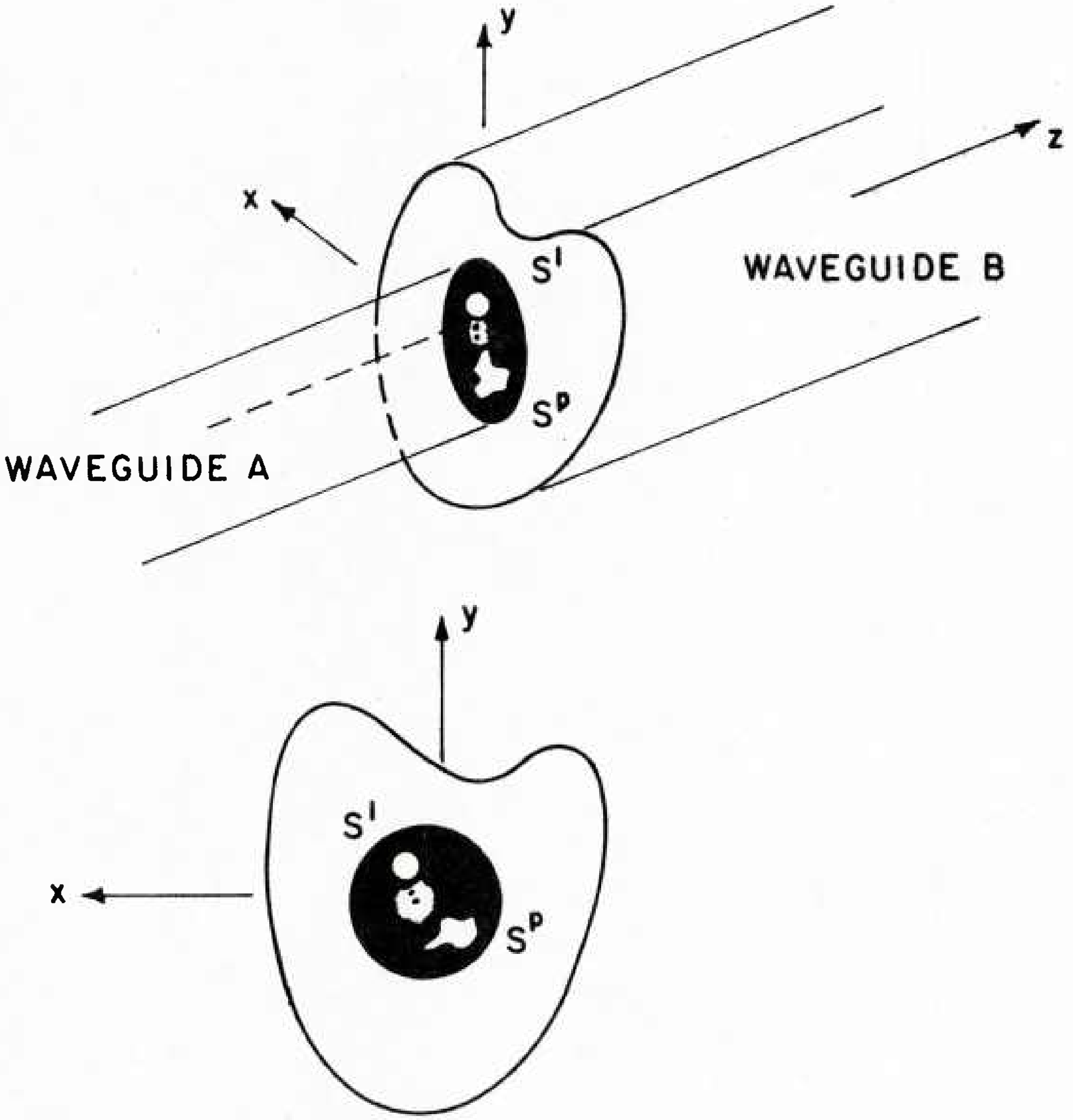

Figure 1. Waveguides A and B opening into each other through $S^1, S^2, \ldots, S^p$.

All the modes TE and TM to z are included in the summation. In (1), c_i, a_i, and b_i, are the amplitudes of the ith incident, reflected, and transmitted modes, respectively. γ_{ai} and η_{ai} are, respectively, the modal propagation constant and characteristic admittance of the ith mode in waveguide A:

$$\gamma_{ai} = \begin{cases} j\beta_i = j\kappa_a\sqrt{1 - \left(\frac{\lambda_a}{\lambda_{ai}}\right)^2} & \lambda_a < \lambda_{ai} \\ \alpha_i = \kappa_{ai}\sqrt{1 - \left(\frac{\lambda_{ai}}{\lambda_a}\right)^2} & \lambda_a > \lambda_{ai} \end{cases} \tag{2}$$

$$\eta_{ai} = \begin{cases} \dfrac{\gamma_{ai}}{j\omega\mu_a} & \text{for TE to z modes} \\ \dfrac{j\omega\varepsilon_a}{\gamma_{ai}} & \text{for TM to z modes.} \end{cases} \tag{3}$$

Here, κ_a is the wave number of the waveguide medium, κ_{ai} is the ith mode cutoff wave number, and λ_a and λ_{ai} are the corresponding wave lengths. The parameters for waveguide B, γ_{bi} and η_{bi}, are similarly defined. Finally, the modal vectors, $\underline{e}_{ai}$ in waveguide A and $\underline{e}_{bi}$ in waveguide B, are assumed real and so normalized that

$$\int_Q \underline{e}_{qi} \cdot \underline{e}_{qj}\, ds = \begin{cases} 1 & i = j \\ 0 & i \neq j\ . \end{cases} \tag{4}$$

The integration in (4) is taken over the cross section of waveguide $Q \in \{A,B\}$, and $q \in \{a,b\}$ is such that $(Q,q) = (A,a)$ or (B,b).

For a complete field solution, the amplitudes a_i and b_i of the various modes in the two waveguides are to be determined. This can be accomplished with the help of a field equivalence theorem.

Let the exciting field be incident in waveguide A while S^m are covered by perfect conductors. This field, sometimes referred to as the generator field, is denoted $(\underline{E}^g, \underline{H}^g)$. The field equivalence theorem [2, Section 3-5] states that the field in waveguide A is identical with $(\underline{E}^g, \underline{H}^g)$ plus the field produced by the magnetic current sheet $\underline{M} = \bigcup_{m=1}^{p} \underline{M}^m$ where

$$\underline{M}^m = \underline{z} \times \underline{E}_t \qquad \text{on } S^m \tag{5}$$

while each S^m is covered by a perfect conductor. The field in waveguide B is then identical with the field produced by the magnetic current $-\underline{M}$ while each S^m is covered by a perfect conductor. Figure 2 shows the equivalent situations.

The z-transverse field produced by $\underline{M}$ in waveguide A, denoted $(\underline{E}_a(\underline{M}), \underline{H}_a(\underline{M}))$, and that produced in waveguide B by $-\underline{M}$, denoted $(\underline{E}_b(-\underline{M}), \underline{H}_b(-\underline{M}))$, have the same form as (1), except that there is no exciting field. Thus, the total z-transverse field in both waveguides is given by

$$\underline{E}_t = \begin{cases} \underline{E}_t^g + \underline{E}_a(\underline{M}) = \sum_i c_i e^{-\gamma_{ai} z} \underline{e}_{ai} - \sum_i c_i e^{\gamma_{ai} z} \underline{e}_{ai} + \sum_i d_i e^{\gamma_{ai} z} \underline{e}_{ai} & z < 0 \\ \\ \underline{E}_b(-\underline{M}) = \sum_i b_i e^{-\gamma_{bi} z} \underline{e}_{bi} & z > 0 \end{cases} \tag{6}$$

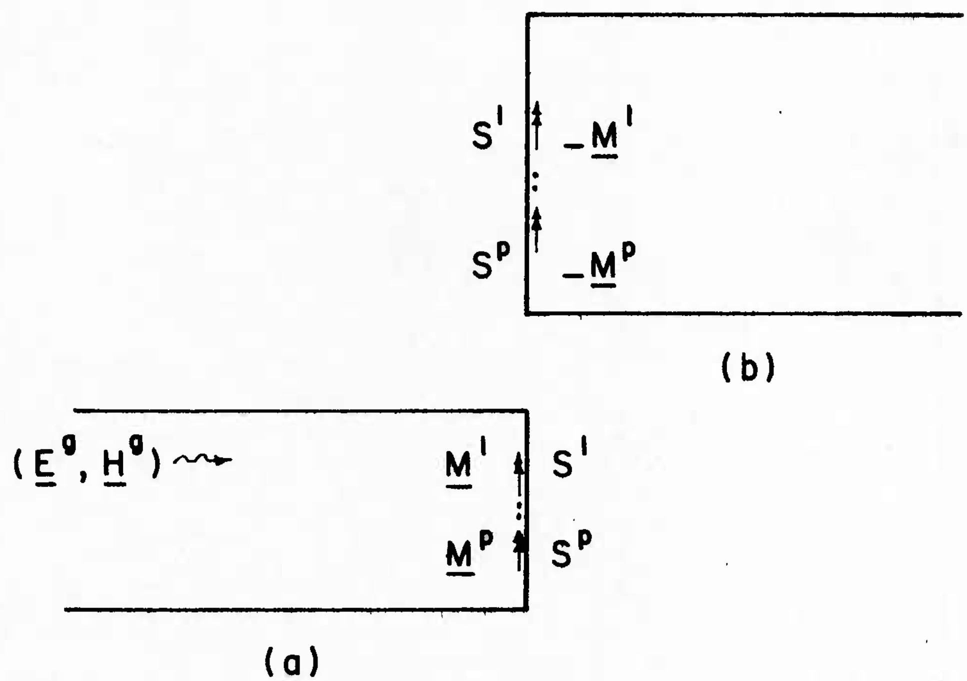

Figure 2. (a) The equivalent situation for waveguide A. $\underline{M}^m$ exists only on S^m.

(b) The equivalent situation for waveguide B. $-\underline{M}^m$ exists only on S^m.

$$
\underline{H}_t = \begin{cases} \underline{H}_t^g + \underline{H}_a(\underline{M}) = \sum_i c_i \eta_{ai} e^{-\gamma_{ai} z} \underline{z} \times \underline{e}_{ai} + \sum_i c_i \eta_{ai} e^{\gamma_{ai} z} \underline{z} \times \underline{e}_{ai} \\ \qquad - \sum_i d_i \eta_{ai} e^{\gamma_{ai} z} \underline{z} \times \underline{e}_{ai} & z < 0 \\ \underline{H}_b(-\underline{M}) = \sum_i b_i \eta_{bi} e^{-\gamma_{bi} z} \underline{z} \times \underline{e}_{bi} & z > 0. \end{cases}
$$

In (6), c_i, d_i, and b_i are, respectively, the amplitudes of the ith incident mode, the ith mode produced in waveguide A by $\underline{M}$, and the ith mode produced in waveguide B by $-\underline{M}$. It then follows from (5) and (6) that

$$
\underline{M} = \begin{cases} \sum_i d_i \, \underline{z} \times \underline{e}_{ai} & \text{on } A \Big|_{z=0} \\ \sum_i b_i \, \underline{z} \times \underline{e}_{bi} & \text{on } B \Big|_{z=0} \end{cases} . \tag{7}
$$

The placement of magnetic current sheets $\underline{M}^m$ in waveguide A and $-\underline{M}^m$ in waveguide B over each S^m therefore ensures the continuity of $\underline{E}_t$ across the apertures. The continuity of $\underline{H}_t$, however, requires that

$$
2 \sum_i c_i \eta_{ai} \, \underline{z} \times \underline{e}_{ai} = \sum_i d_i \eta_{ai} \, \underline{z} \times \underline{e}_{ai} + \sum_i b_i \eta_{bi} \, \underline{z} \times \underline{e}_{bi}
$$

$$
\text{on } S^m, \; 1 \le m \le p \tag{8}
$$

which is, together with (7), the equation determining the amplitudes d_i and b_i.

2. The Generalized Network Representation

An exact solution of (7) and (8) for the amplitudes d_i and b_i, and consequently the complete field solution, can rarely be obtained. However, only a representation of the apertures that describes their effect on the modes of the waveguides, not the amplitudes of these modes, is usually all that is needed. In this section, a representation in terms of two generalized networks is derived.

Since the set $\{\underline{e}_{qi} \mid i = 1,2,\ldots\}$ is complete [9, Section 5-6], a finite subset of the lower order modes can be used to approximate the field in waveguide Q. Thus, (7) and (8) become

$$\underline{M} = \begin{cases} \displaystyle\sum_{i=1}^{\ell_a} d_i \, \underline{z} \times \underline{e}_{ai} & \text{on } A\Big|_{z=0} \\ \displaystyle\sum_{i=1}^{\ell_b} b_i \, \underline{z} \times \underline{e}_{bi} & \text{on } B\Big|_{z=0} \end{cases} \tag{9}$$

$$2 \sum_{i=1}^{\ell_a} c_i \eta_{ai} \, \underline{z} \times \underline{e}_{ai} = \sum_{i=1}^{\ell_a} d_i \eta_{ai} \, \underline{z} \times \underline{e}_{ai} + \sum_{i=1}^{\ell_b} b_i \eta_{bi} \, \underline{z} \times \underline{e}_{bi}$$

$$\text{on } S^m, \ 1 \le m \le p \tag{10}$$

where ℓ_a and ℓ_b are, respectively, the number of modes used in the modal expansion of the fields in waveguides A and B. Here, d_i and b_i are no longer the actual mode amplitudes. Only in the limit, as ℓ_a and ℓ_b go to infinity, they become so. However, for sufficiently large ℓ_a and ℓ_b, it can be assumed that d_i and b_i in (9) and (10) represent the actual mode amplitudes, while equality still holds there.

Let $\{\underline{M}_j^m | 1 \leq j \leq p^m\}$ be a set of real vectors, and put

$$\underline{M}^m = \sum_{j=1}^{p^m} V_j^m \underline{M}_j^m \tag{11}$$

where V_j^m are complex coefficients to be determined. Substituting (11) into (9), it becomes

$$\sum_{j=1}^{p^m} V_j^m \underline{M}_j^m = \begin{cases} \sum_{i=1}^{\ell_a} d_i \, \underline{z} \times \underline{e}_{ai} & \text{on } A\big|_{z=0} \\ \\ \sum_{i=1}^{\ell_b} b_i \, \underline{z} \times \underline{e}_{bi} & \text{on } B\big|_{z=0} \end{cases} . \tag{12}$$

Put

$$w_{qk} = \int_{Q|_{z=0}} \underline{M} \cdot \underline{z} \times \underline{e}_{qk} \, ds \; . \tag{13}$$

It then follows from (4) and (12) that

$$\begin{aligned} w_{ak} &= \sum_{m=1}^{p} \sum_{j=1}^{p^m} V_j^m \int_{S^m} \underline{M}_j^m \cdot \underline{z} \times \underline{e}_{ak} ds = d_k \\ &= \sum_{i=1}^{\ell_b} b_i \int_{A \cap B} \underline{z} \times \underline{e}_{bi} \cdot \underline{z} \times \underline{e}_{ak} \, ds, \quad 1 \leq k \leq \ell_a \end{aligned} \tag{14}$$

$$\begin{aligned} w_{bk} &= \sum_{m=1}^{p} \sum_{j=1}^{p^m} V_j^m \int_{S^m} \underline{M}_j^m \cdot \underline{z} \times \underline{e}_{bk} \, ds \\ &= \sum_{i=1}^{\ell_a} d_i \int_{A \cap B} \underline{z} \times \underline{e}_{ai} \cdot \underline{z} \times \underline{e}_{bk} \, ds = b_k, \quad 1 \leq k \leq \ell_b . \end{aligned} \tag{15}$$

Finally, scalarly multiplying (10) by $\underline{M}_j^m$ and integrating over S^m, it becomes

$$2 \sum_{i=1}^{\ell_a} c_i \eta_{ai} \int_{S^m} \underline{M}_j^m \cdot \underline{z} \times \underline{e}_{ai} ds = \sum_{i=1}^{\ell_a} d_i \eta_{ai} \int_{S^m} \underline{M}_j^m \cdot \underline{z} \times \underline{e}_{ai} \, ds$$

$$+ \sum_{i=1}^{\ell_b} b_i \eta_{bi} \int_{S^m} \underline{M}_j^m \cdot \underline{z} \times \underline{e}_{bi} ds, \quad 1 \leq m \leq p, \; 1 \leq j \leq p^m . \tag{16}$$

In matrix form, (14), (15), and (16) become

$$\vec{w}_a = H_a \vec{V} = \vec{d} = H^T \vec{b} \tag{17}$$

$$\vec{w}_b = H_b \vec{V} = H\vec{d} = \vec{b} \tag{18}$$

$$2 \, H_a^T Y_{oa} \vec{c} = H_a^T Y_{oa} \vec{d} + H_b^T Y_{ob} \vec{b} . \tag{19}$$

Here, T denotes matrix transpose, $\vec{w}$ is the vector

$$\vec{w}_q = [w_{qj}]_{\ell_q \times 1} \tag{20}$$

$\vec{V}$ is the p segment vector whose mth segment is the vector

$$\vec{V}^m = [V_j^m]_{p^m \times 1} \tag{21}$$

$\vec{c}$, $\vec{d}$, and $\vec{b}$ are the vectors

$$\vec{c} = [c_j]_{\ell_a \times 1} \tag{22}$$

$$\vec{d} = [d_j]_{\ell_a \times 1} \tag{23}$$

$$\vec{b} = [b_j]_{\ell_b \times 1} \tag{24}$$

H_q is the p block row matrix whose mth block is the matrix

$$H_q^m = [H_{qjj}^m]_{\ell_q \times p^m} = [\int_{S^m} \underline{z} \times \underline{e}_{qi} \cdot \underline{M}_j \, ds]_{\ell_q \times p^m} \tag{25}$$

Y_{oq} is the diagonal matrix

$$Y_{oq} = [Y_{oqii}]_{\ell_q \times \ell_q} = [\eta_{qi}]_{\ell_q \times \ell_q} \tag{26}$$

and H is the matrix

$$H = [H_{ij}]_{\ell_b \times \ell_a} = [\int_{A \cap B} \underline{z} \times \underline{e}_{bi} \cdot \underline{z} \times \underline{e}_{aj} ds]_{\ell_b \times \ell_a} . \tag{27}$$

Since H_a, H_b, and H depend only on the sets of waveguide modes used in the modal expansion, the set $\bigcup_{m=1}^{p} \bigcup_{j=1}^{p^m} \{\underline{M}_j^m\}$ of magnetic currents, and the shape of the apertures, it readily follows from (17) and (18) that

$$H_b = HH_a \tag{28}$$

$$H^T H = U \tag{29}$$

$$HH^T = U . \tag{30}$$

U is the identity matrix of order ℓ_a in (29), and of order ℓ_b in (30). Using (17) and (18), (19) becomes

$$(\bar{\bar{Y}}_a + \bar{\bar{Y}}_b)\vec{V} = \vec{I} \tag{31}$$

where

$$\bar{\bar{Y}}_q = H_q^T \, Y_{oq} \, H_q \tag{32}$$

$$\vec{I} = 2 \, H_a^T Y_{oa} \vec{c} . \tag{33}$$

By (31), the generalized network representation of the apertures is finally obtained. This representation consists of two networks $\bar{\bar{Y}}_a$ and $\bar{\bar{Y}}_b$ in parallel with the current source $\vec{I}$, as (31) readily indicates. Figure 3 depicts such a representation. In the following two sections, the generalized network representation is shown to obey the two basic network laws: the conservation of complex power law and the reciprocity law.

3. <u>The conservation of Complex Power Law</u>

The complex power transmitted through the apertures into waveguide B is basically

$$P_{tr} = \int_{B|_{z=0}} \underline{E} \times \underline{H}^* \cdot \underline{z}\, ds \tag{34}$$

where * denotes complex conjugate. Substituting from (5) and (6), and using (13), (34) becomes

$$P_{tr} = \sum_{i=1}^{\ell_b} b_i^* \, \eta_{bi}^* \, w_{bi} \tag{35}$$

In matrix form, (35) becomes

$$P_{tr} = \vec{b}^H \, Y_{ob}^* \, \vec{w}_b \quad . \tag{36}$$

In (36), H denotes matrix Hermitian. It then follows from (18) and (32) that

$$P_{tr} = \vec{b}^H Y_{ob}^* \vec{b} = \vec{V}^H \, H_b^H Y_{ob}^* H_b \vec{V} = \vec{V}^H \bar{\bar{Y}}_b^H \, \vec{V} \; . \tag{37}$$

Thus, the complex power transmitted into waveguide B is equal to the

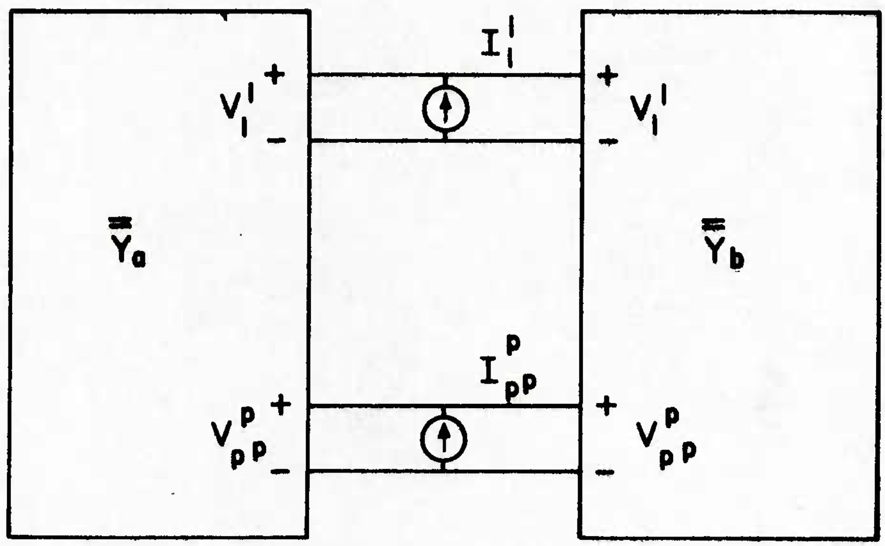

Figure 3. The generalized network representation of the apertures.

complex power at the terminals of network $\bar{\bar{Y}}_b$.

The total complex power entering the apertures from waveguide A is given by

$$P_{in} = \int_{A|_{z=0}} \underline{E} \times \underline{H}^* \cdot \underline{z}\, ds \quad . \tag{38}$$

Substituting from (5) and (6), and using (13), (38) becomes

$$P_{in} = \sum_{i=1}^{\ell_a} (2c_i^* - d_i^*)\eta_{ai}^* w_{ai} \quad . \tag{39}$$

In matrix form

$$P_{in} = 2\vec{c}^H Y_{oa}^* \vec{w}_a - \vec{d}^H Y_{oa}^* \vec{w}_a \quad . \tag{40}$$

It then follows from (17), (32), and (33) that

$$P_{in} = 2\vec{c}^H Y_{oa}^* \vec{d} - \vec{d}^H Y_{oa}^* \vec{d} = 2\vec{c}^H Y_{oa}^* H_a \vec{V} - \vec{V}^H H_a^H Y_{oa}^* H_a \vec{V}$$

$$= \vec{I}^H \vec{V} - \vec{V}^H \bar{\bar{Y}}_a^H \vec{V} \quad . \tag{41}$$

Thus, the total complex power entering the apertures from waveguide A is equal to the complex power supplied by the source $\vec{I}$ minus that at the terminals of network $\bar{\bar{Y}}_a$.

The conservation of power law states that the total complex power in any network must be zero, or

$$P_{tr} - P_{in} \equiv 0 \quad . \tag{42}$$

This is guaranteed because of (31). The conservation of complex power law holds if and only if the complex power flow across the apertures is continuous.

4. The Reciprocity Law

Let a multi-mode field be incident in waveguide B. The total z-transverse field in both waveguides is then

$$\underline{E}_t' = \begin{cases} \sum_i d_i' e^{\gamma_{ai} z} \underline{e}_{ai} & z < 0 \\ \sum_i c_i' e^{\gamma_{bi} z} \underline{e}_{bi} + \sum_i a_i' e^{-\gamma_{bi} z} \underline{e}_{bi} & z > 0 \end{cases}$$

$$\underline{H}' = \begin{cases} -\sum_i d_i' \eta_{ai} e^{\gamma_{ai} z} \underline{z} \times \underline{e}_{ai} & z < 0 \\ -\sum_i c_i' \eta_{bi} e^{\gamma_{bi} z} \underline{z} \times \underline{e}_{bi} + \sum_i a_i' \eta_{bi} e^{-\gamma_{bi} z} \underline{z} \times \underline{e}_{bi} & z > 0 \end{cases} \qquad (43)$$

where c_i', a_i', and d_i' are, respectively, the amplitudes of the ith incident, reflected, and transmitted modes.

The generator field $(\underline{E}'^g, \underline{H}'^g)$ is the field that would exist if the exciting field was incident in waveguide B while all S^m were covered by perfect conductors. By the field equivalence theorem, the z-transverse field in waveguide B is identical with $(\underline{E}_t'^g, \underline{H}_t'^g)$ plus the z-transverse field $(\underline{E}_b'(-\underline{M}'), \underline{H}_b'(-\underline{M}'))$ produced by the magnetic current sheet $-\underline{M}' = \bigcup_{m=1}^{p} - \underline{M}^m$ where

$$\underline{M}'^m = \underline{z} \times \underline{E}_t' \qquad \text{on } S^m \qquad (44)$$

while each S^m is covered by a perfect conductor. The z-transverse field in waveguide A is then identical with the z-transverse field $(\underline{E}'_a(\underline{M}'),\ \underline{H}'_a(\underline{M}'))$ produced by the magnetic current $\underline{M}'$ while each S^m is covered by a perfect conductor.

The total z-transverse field in both waveguides is then

$$\underline{E}'_t = \begin{cases} \underline{E}'_a(\underline{M}') = \sum_i d'_i\, e^{\gamma_{ai} z}\, \underline{e}_{ai} & z < 0 \\ \underline{E}'^g_t + \underline{E}'_b(-\underline{M}') = \sum_i c'_i e^{\gamma_{bi} z}\, \underline{e}_{bi} - \sum_i c'_i e^{-\gamma_{bi} z}\, \underline{e}_{bi} + \sum_i b'_i e^{-\gamma_{bi} z}\, \underline{e}_{bi} & z > 0 \end{cases}$$

$$\underline{H}'_t = \begin{cases} \underline{H}'_a(\underline{M}') = -\sum_i d'_i \eta_{ai}\, e^{\gamma_{ai} z}\, \underline{z} \times \underline{e}_{ai} & z < 0 \\ \underline{H}'^g_t + \underline{H}'_b(-\underline{M}') = -\sum_i c'_i \eta_{bi}\, e^{\gamma_{bi} z}\, \underline{z} \times \underline{e}_{bi} - \sum_i c'_i \eta_{bi} e^{-\gamma_{bi} z}\, \underline{z} \times \underline{e}_{bi} & \\ \qquad + \sum_i b'_i \eta_{bi}\, e^{-\gamma_{bi} z}\, \underline{z} \times \underline{e}_{bi} & z > 0\ . \end{cases} \tag{45}$$

In (45), c'_i, d'_i, and b'_i are the amplitudes of the ith incident mode, the ith mode produced by $\underline{M}'$ in waveguide A, and the ith mode produced by $-M'$ in waveguide B, respectively. The continuity of $\underline{E}'_t$ across the apertures implies that

$$\underline{M}' = \begin{cases} \sum_i d'_i\, \underline{z} \times \underline{e}_{ai} & \text{on } A \Big|_{z=0} \\ \sum_i b'_i\, \underline{z} \times \underline{e}_{bi} & \text{on } B \Big|_{z=0} \end{cases} \tag{46}$$

whereas for that of $\underline{H}'_t$

$$2 \sum_i c_i' \eta_{bi} \underline{z} \times \underline{e}_{bi} = \sum_i d_i' \eta_{ai} \underline{z} \times \underline{e}_{ai} + \sum_i b_i' \eta_{bi} \underline{z} \times \underline{e}_{bi}$$

$$\text{on } S^m, \quad 1 \leq m \leq p \qquad (47)$$

must be satisfied.

Let a finite subset of the lower modes in waveguide Q be used to approximate the field there. The cardinality of this set is set equal to ℓ_q. Put

$$\underline{M}'^m = \sum_{j=1}^{p^m} V_j'^m \underline{M}_j^m \qquad (48)$$

$$w'_{qi} = \int_{Q|_{z=0}} \underline{M}' \cdot \underline{z} \times \underline{e}_{qi} \, ds \qquad (49)$$

where $V_j'^m$ are complex coefficients to be determined. This situation is analogous to that in Section 2, and can similarly be treated. Thus, in analogy with (17), (18), and (19), one obtains

$$\vec{w}'_a = H_a \vec{V}' = \vec{d}' = H^T \vec{b}' \qquad (50)$$

$$\vec{w}'_b = H_b \vec{V}' = H\vec{d}' = \vec{b}' \qquad (51)$$

$$2H_b^T Y_{ob} \vec{c}' = H_a^T Y_{oa} \vec{d}' + H_b^T Y_{ob} \vec{b}' \qquad (52)$$

Here, $\vec{w}'_q$, $\vec{V}'$, $\vec{c}'$, $\vec{d}'$, and $\vec{b}'$ are defined as are their counterparts $\vec{w}_q$, $\vec{V}$, $\vec{c}$, $\vec{d}$, and $\vec{b}$, respectively.

Using (50) and (51), (52) becomes

$$(\bar{\bar{Y}}_a + \bar{\bar{Y}}_b) \vec{V}' = \vec{I}' \qquad (53)$$

where

$$\vec{I}' = 2H_b^T Y_{ob}\vec{c} \; . \tag{54}$$

The generalized network representations of the apertures corresponding to (53) is similar to that shown in Figure 3, except for $\vec{V}'$ and $\vec{I}'$ replacing $\vec{V}$ and $\vec{I}$, respectively. The reciprocity law then states that

$$\vec{V}^T\vec{I}' = \vec{V}'^T\vec{I} \; . \tag{55}$$

This can be shown with the help of the reciprocity theorem.

The reciprocity theorem [2, Section 3-8] states that

$$\int_W (\underline{E}^1 \times \underline{H}^2 - \underline{E}^2 \times \underline{H}^1) \cdot \underline{n} \, ds = 0 \tag{56}$$

where W is the closed surface enclosing the volume containing the apertures, $(\underline{E}^1, \underline{H}^1)$ and $(\underline{E}^2, \underline{H}^2)$ are source-free fields within this volume, and $\underline{n}$ is the outward unit vector normal to W. Let W be the surface consisting of all the metallic walls between the cross sections of waveguide A at $z = z_1$ and waveguide B at $z = z_2$, for some $z_1 < 0$ and $z_2 > 0$, and these two waveguide cross sections, and let $(\underline{E}^1, \underline{H}^1)$ and $(\underline{E}^2, \underline{H}^2)$ be the fields whose transverse components are given by (6) and (45), respectively. Substituting in (56) and using (4), (56) becomes

$$2 \sum_{i=1}^{\ell_a} c_i \eta_{ai} d_i' = 2 \sum_{i=1}^{\ell_b} c_i' \eta_{bi} b_i \tag{57}$$

or in matrix form

$$2\vec{d}'^T Y_{oa}\vec{c} = 2\vec{b}^T Y_{ob}\vec{c}' \; . \tag{58}$$

The reciprocity law then follows from (18), (33), (50), (54), and (58). The reciprocity law holds if and only if the whole structure is reciprocal.

5. The Scattering Matrix

Another representation of the apertures is in terms of their scattering matrix. Following Montgomery et al. [10, Section 5-14], the scattering matrix is defined as

$$S = \begin{bmatrix} S_{aa} & S_{ab} \\ S_{ba} & S_{bb} \end{bmatrix} \tag{59}$$

where the ijth element of S_{qo} is the amplitude of the ith mode in waveguide Q due to the jth mode of unit amplitude incident in waveguide O (O ε {A,B}).

Put

$$\vec{a} = [a_j]_{\ell_a \times 1} \tag{60}$$

$$\vec{a}' = [a_j']_{\ell_b \times 1} . \tag{61}$$

It then follows from (1), (6), (22), (23), (43), and (45) that

$$\vec{a} = \vec{d} - \vec{c} \tag{62}$$

$$\vec{a}' = \vec{b}' - \vec{c}' . \tag{63}$$

The scattering submatrices S_{aa}, S_{ba}, S_{ab}, and S_{bb} are then given by

$$\vec{a} = S_{aa}\vec{c} \tag{64}$$

$$\vec{b} = S_{ba}\vec{c} \tag{65}$$

$$\vec{d}' = S_{ab}\vec{c}' \tag{66}$$

$$\vec{a}' = S_{bb}\vec{c}' \ . \tag{67}$$

The submatrices of S can be deduced from the analysis in Sections 2 and 4, almost immediately. Using (17), (31), and (33), (62) becomes

$$\vec{a} = (2\ H_a(\bar{\bar{Y}}_a + \bar{\bar{Y}}_b)^{-1}\ H_a^T Y_{oa} - U)\vec{c} \ . \tag{68}$$

Consequently

$$S_{aa} = 2\ H_a(\bar{\bar{Y}}_a + \bar{\bar{Y}}_b)^{-1}\ H_a^T Y_{oa} - U \ . \tag{69}$$

Similarly, from (18), (31), (33), and (65), it follows that

$$S_{ba} = 2\ H_b(\bar{\bar{Y}}_a + \bar{\bar{Y}}_b)^{-1}\ H_a^T Y_{oa} \tag{70}$$

or, on using (28) and (69),

$$S_{ba} = H(S_{aa} + U). \tag{71}$$

The reciprocity law can be used to determine S_{ab}. Substituting (65) and (66) into (58), it becomes

$$\vec{c}'^T\ S_{ab}^T\ Y_{oa}\vec{c} = \vec{c}^T\ S_{ba}^T\ Y_{ob}\vec{c}' \quad . \tag{72}$$

Thus

$$S_{ab} = Y_{oa}^{-1}\ S_{ba}^T\ Y_{ob} \tag{73}$$

since (72) holds for all $\vec{c}$ and $\vec{c}'$.

Finally, using (51), (53), and (54), (63) becomes

$$\vec{a}' = (2\, H_b(\bar{\bar{Y}}_a + \bar{\bar{Y}}_b)^{-1}\, H_b^T Y_{ob} - U)\, \vec{c}' \,. \tag{74}$$

Consequently

$$S_{bb} = 2\, H_b(\bar{\bar{Y}}_a + \bar{\bar{Y}}_b)^{-1}\, H_b^T\, Y_{ob} - U \tag{75}$$

or

$$S_{bb} = HS_{ab} - U \tag{76}$$

as follows from (51), (63), (66), and (67).

Let the multi-mode field be incident on the apertures simultaneously in waveguides A and B. The total z-transverse field in both waveguides is readily found from (1) and (43) as

$$\underline{E}_t = \begin{cases} \sum\limits_i c_i e^{-\gamma_{ai} z} \underline{e}_{ai} + \sum\limits_i a_i e^{\gamma_{ai} z} \underline{e}_{ai} + \sum\limits_i d_i' e^{\gamma_{ai} z} \underline{e}_{ai} & z < 0 \\[2ex] \sum\limits_i b_i e^{-\gamma_{bi} z} \underline{e}_{bi} + \sum\limits_i c_i' e^{\gamma_{bi} z} \underline{e}_{bi} + \sum\limits_i a_i' e^{-\gamma_{bi} z} \underline{e}_{bi} & z > 0 \end{cases} \tag{77}$$

$$\underline{H}_t = \begin{cases} \sum\limits_i c_i \eta_{ai} e^{-\gamma_{ai} z} \underline{z} \times \underline{e}_{ai} - \sum\limits_i a_i \eta_{ai} e^{\gamma_{ai} z} \underline{z} \times \underline{e}_{ai} \\ \qquad - \sum\limits_i d_i' \eta_{ai} e^{\gamma_{ai} z} \underline{z} \times \underline{e}_{ai} & z < 0 \\[2ex] \sum\limits_i b_i \eta_{bi} e^{-\gamma_{bi} z} \underline{z} \times \underline{e}_{bi} - \sum\limits_i c_i' \eta_{bi} e^{\gamma_{bi} z} \underline{z} \times \underline{e}_{bi} \\ \qquad + \sum\limits_i a_i' \eta_{bi} e^{-\gamma_{bi} z} z \times \underline{e}_{bi} & z > 0 \,. \end{cases}$$

The complex power to the left of the apertures is then given by

$$P_{0_-} = (\vec{c} + \vec{a} + \vec{d}')^H\, Y^*_{oa}(\vec{c} - \vec{a} - \vec{d}') \tag{78}$$

whereas that to the right of the apertures is

$$P_{0_+} = (\vec{b} + \vec{c}' + \vec{a}')^H\, Y^*_{ob}(\vec{b} - \vec{c}' + \vec{a}') \ . \tag{79}$$

Here, as before, a finite subset of the lower order modes of cardinality ℓ_q is used to approximate the field in waveguide Q. By the conservation of complex power law, P_{0_-} and P_{0_+} must be equal. Put

$$Y = \begin{bmatrix} Y_{oa} & \\ & Y_{ob} \end{bmatrix} \tag{80}$$

$$\vec{C} = \begin{bmatrix} \vec{c} \\ \vec{c}' \end{bmatrix} \ . \tag{81}$$

Then, using (59), (64), (65), (66), and (67),

$$\vec{C}^H S^H Y^* S\, \vec{C} + \vec{C}^H (Y^* S - S^H Y^*)\vec{C} = \vec{C}^H Y^* \vec{C} \ . \tag{82}$$

Since $\vec{C}$ is completely arbitrary, (82) gives

$$S^H Y^* S + (Y^* S - S^H Y^*) = Y^* \tag{83}$$

Although the whole structure is both reciprocal and lossless, the scattering matrix is neither symmetric nor unitary.

6. Inductive Windows in a Rectangular Waveguide

Consider a system of windows W^1, $W^2,\ldots,W^p$ located in the $z=0$ plane in a rectangular waveguide. These windows are assumed uniform along the narrow side of the waveguide, i.e., of the inductive type. The waveguide medium is assumed characterized by the real scalar constitutive parameters ε and μ. Figure 4 shows the situation at hand.

Let a TE_{10} to z mode of unit amplitude be incident on the windows from the left. This mode has the field distribution

$$\left.\begin{aligned} \underline{E}^i &= e^{-\gamma_1 z}\,\underline{e}_1 \\ \underline{H}^i &= \eta_1 e^{-\gamma_1 z}\,\underline{z}\times\underline{e}_1 - \frac{1}{j\omega\mu} e^{-\gamma_1 z}\frac{\partial}{\partial x}\,\underline{x}\times\underline{e}_1 \quad . \end{aligned}\right\} \tag{84}$$

In (84)

$$\underline{e}_1 = \sqrt{\frac{2}{ab}}\,\sin\left(\frac{\pi}{a}\,x\right)\underline{y} \tag{85}$$

and the subscript "a" is dropped from γ_1 and η_1. Furthermore, it is assumed that $a < \lambda < 2a$ and $2b < \lambda$ so that only the dominant mode can propagate in the waveguide.

Since the window-waveguide structure is uniform along the y-axis, and since the incident mode has only an E_y component that does not vary with y, the field scattered must have only an E_y component that does not vary with y. The only modes excited in the waveguide are therefore TE_{no} to z modes, since these are the only modes having only an E_y component that does not vary with y [2, Section 4-3]. The field distribution of any such mode is similar to that

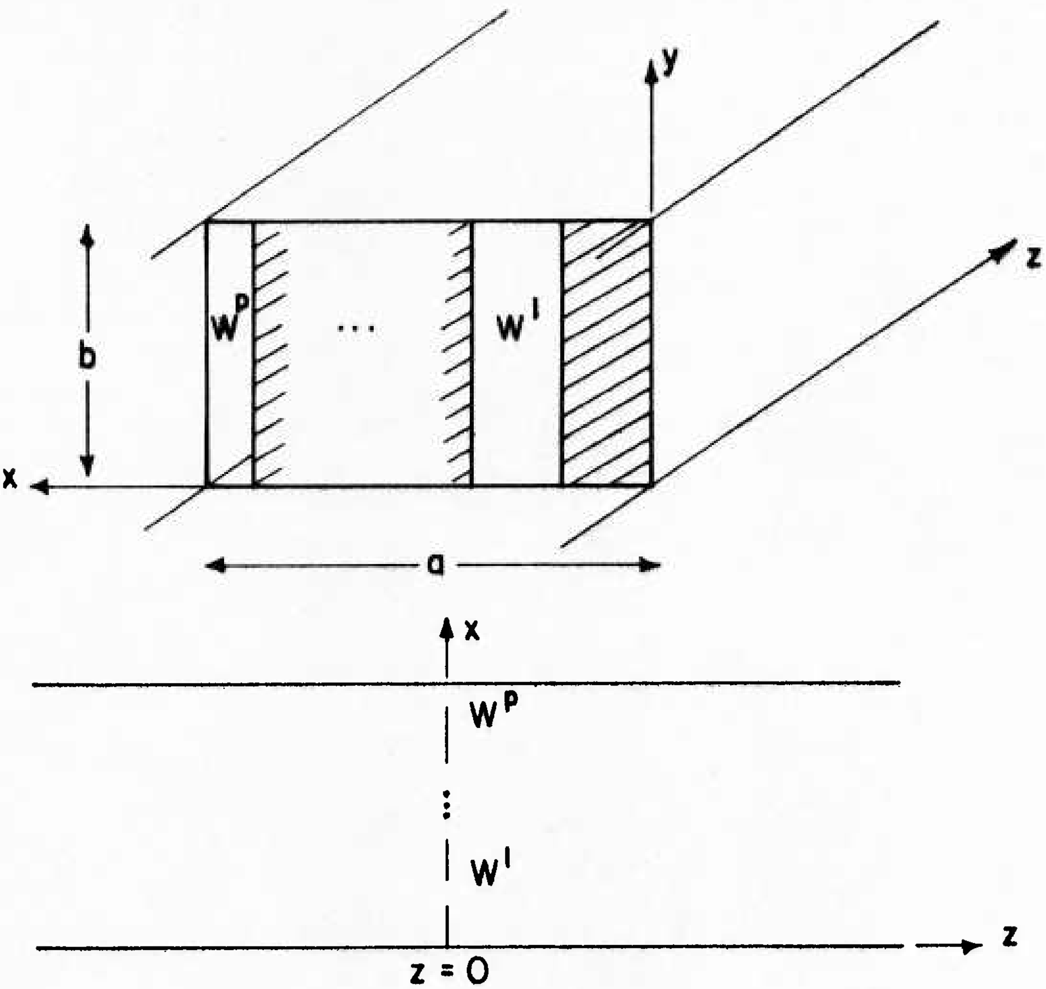

Figure 4. p inductive windows in a rectangular waveguide.

of the dominant mode, viz.

$$\left.\begin{aligned} \underline{E}_n &= e^{\pm\gamma_n z}\, \underline{e}_n \\ \underline{H}_n &= \mp\, \eta_n\, e^{\pm\gamma_n z}\, \underline{z} \times \underline{e}_n - \frac{1}{j\omega\mu}\, e^{\pm\gamma_n z}\, \frac{\partial}{\partial x}\, \underline{x} \times \underline{e}_n \end{aligned}\right\} \tag{86}$$

where

$$\underline{e}_n = \sqrt{\frac{2}{ab}}\, \sin\left(\frac{n\pi}{a}\, x\right)\, \underline{y} \tag{87}$$

and

$$\gamma_n = \sqrt{\left(\frac{n\pi}{a}\right)^2 - \kappa^2}\quad . \tag{88}$$

The change of subscripts from i to n is for later convenience.

The total z-transverse field has only a y-component of electric field that does not vary with y. It then follows from (5) that each $\underline{M}^m$ has only an x-component that does not vary with y:

$$\underline{M}^m = M^m(x)\underline{x} \qquad \text{on } W^m\ . \tag{89}$$

The generalized network representation of the windows can be obtained as indicated in Section 2. Here, however, the whole set of modes is used, which calls for some minor changes in the formulas there.

Since the dominant mode is the only incident mode, $\vec{c}$ now becomes

$$\vec{c} = [\delta_{1n}]_{\infty \times 1}\ . \tag{90}$$

In (90), δ_{un} is the Kronecker delta function

$$\delta_{un} = \begin{cases} 1 & u = n \\ 0 & u \neq n . \end{cases} \tag{91}$$

The vectors $\vec{w}_q$, $\vec{d}$, and $\vec{b}$ are likewise vectors of infinite length. Furthermore, since the mediums on both sides of the windows are identical,

$$\vec{w}_a = \vec{w}_b = \vec{w} \tag{92}$$

$$H_a = H_b = H_o \tag{93}$$

$$Y_{oa} = Y_{ob} = Y_o . \tag{94}$$

H is then the identity matrix of infinite order. Consequently

$$\bar{\bar{Y}}_a = \bar{\bar{Y}}_b = \bar{\bar{Y}} = H_o^T Y_o H_o . \tag{95}$$

The generalized network representation of the windows then becomes

$$\bar{\bar{Y}} \vec{V} = \vec{I} \tag{96}$$

where $\bar{\bar{Y}}$ is the p by p block matrix whose ℓmth block is the matrix

$$\bar{\bar{Y}}^{\ell m} = [\bar{\bar{Y}}_{ij}^{\ell m}]_{p^\ell \times p^m} = [\sum_{n=1}^{\infty} \eta_n \int_{W^\ell} M_i^\ell e_n ds' \int_{W^m} e_n M_j^m ds]_{p^\ell \times p^m} \tag{97}$$

and $\vec{I}$ is the p segment vector whose ℓth segment is the vector

$$\vec{I}^\ell = [I_i^\ell]_{p^\ell \times 1} = [-\eta_1 \int_{W^\ell} M_i^\ell e_1 ds]_{p^\ell \times 1} . \tag{98}$$

This representation is depicted in Figure 5.

Minor changes are also due for the scattering matrix of the windows. The higher order $(n > 1)$ modes excited are evanescent, i.e., decay exponentially with distance from the windows. Thus, at sufficiently large distances, only the dominant mode exists. The scattering matrix can then be defined as

$$S = \begin{bmatrix} a_1 & d_1' \\ b_1 & a_1' \end{bmatrix} . \tag{99}$$

The mode-amplitude vectors there then reduce to scalars, viz.,

$$\vec{g} = \begin{cases} g_1 , & \vec{g} \in \{\vec{a}, \vec{b}, \vec{d}, \vec{a}', \vec{b}', \vec{d}'\} \\ 1 , & \vec{g} \in \{\vec{c}, \vec{c}'\} \end{cases} \tag{100}$$

while H_o becomes the p segment row vector $\vec{h}_o$ whose mth segment is the row vector

$$\vec{h}_o^m = [h_{oj}^m]_{1 \times p^m} = [- \int_{W^m} e_1 \, M_j^m \, ds]_{1 \times p^m} . \tag{101}$$

H is then unity. The scattering parameters are readily found from (69), (71), (73), and (76) as

$$a_1 = a_1' = - 1 - \sum_{m=1}^{p} \sum_{j=1}^{p^m} V_j^m \int_{W^m} e_1 \, M_j^m \, ds \tag{102}$$

$$b_1 = d'_1 = 1 + a_1 = 1 + a_1' . \tag{103}$$

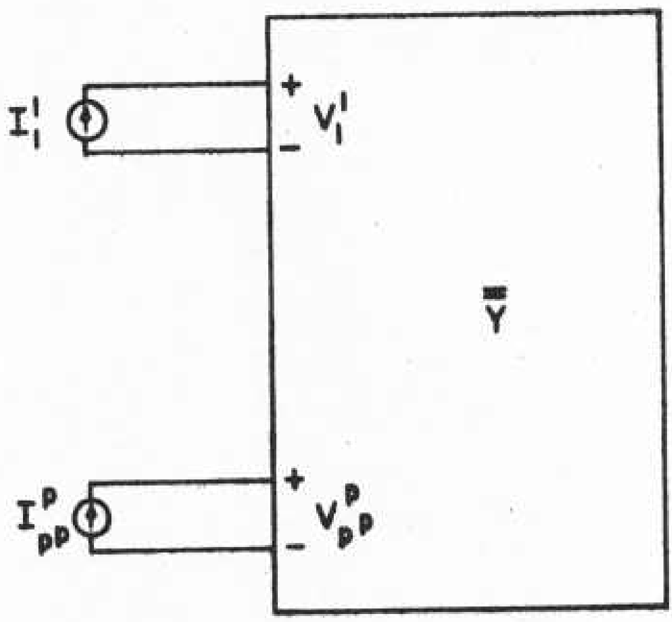

Figure 5. The generalized network representation of the windows.

The scattering matrix is both symmetric and unitary. Notice that the second term in the left-hand side of (82) then accounts for the reactive power basically due to the evanescent modes.

7. Capacitive Windows in a Rectangular Waveguide

Consider now the system of windows $W^1, W^2, \ldots, W^p$ in a rectangular waveguide shown in Figure 6. These windows, as can be seen, are uniform along the broad side of the waveguide, i.e., of the capacitive type. The waveguide medium is assumed characterized by the real scalar constitutive parameters ϵ and μ.

Let a TE_{10} to z mode of unit amplitude be incident on the windows from the left. This mode has the field distribution

$$\left.\begin{aligned} \underline{E}^i &= e^{-\gamma_0 z}\, \underline{e}_0 \\ \underline{H}^i &= \eta_0\, e^{-\gamma_0 z}\, \underline{z} \times \underline{e}_0 - \frac{1}{j\omega\mu}\, e^{-\gamma_0 z} \frac{\partial}{\partial x}\, \underline{x} \times \underline{e}_0 \end{aligned}\right\} \tag{104}$$

where

$$\underline{e}_0 = \sqrt{\frac{2}{ab}} \sin\left(\frac{\pi}{a} x\right) \underline{y}. \tag{105}$$

Here, the index of the dominant mode is "0" rather than "1" as in (84) and (85). It is still assumed, however, that $a < \lambda < 2a$ and $2b < \lambda$ so that only the dominant mode can propagate in the waveguide.

Since the window-waveguide structure is uniform along the x-axis, and since the incident mode has an H_x component that varies as $\sin\left(\frac{\pi}{a} x\right)$ and no x-component of electric field, so must be the scattered field. The only modes excited in the waveguide are

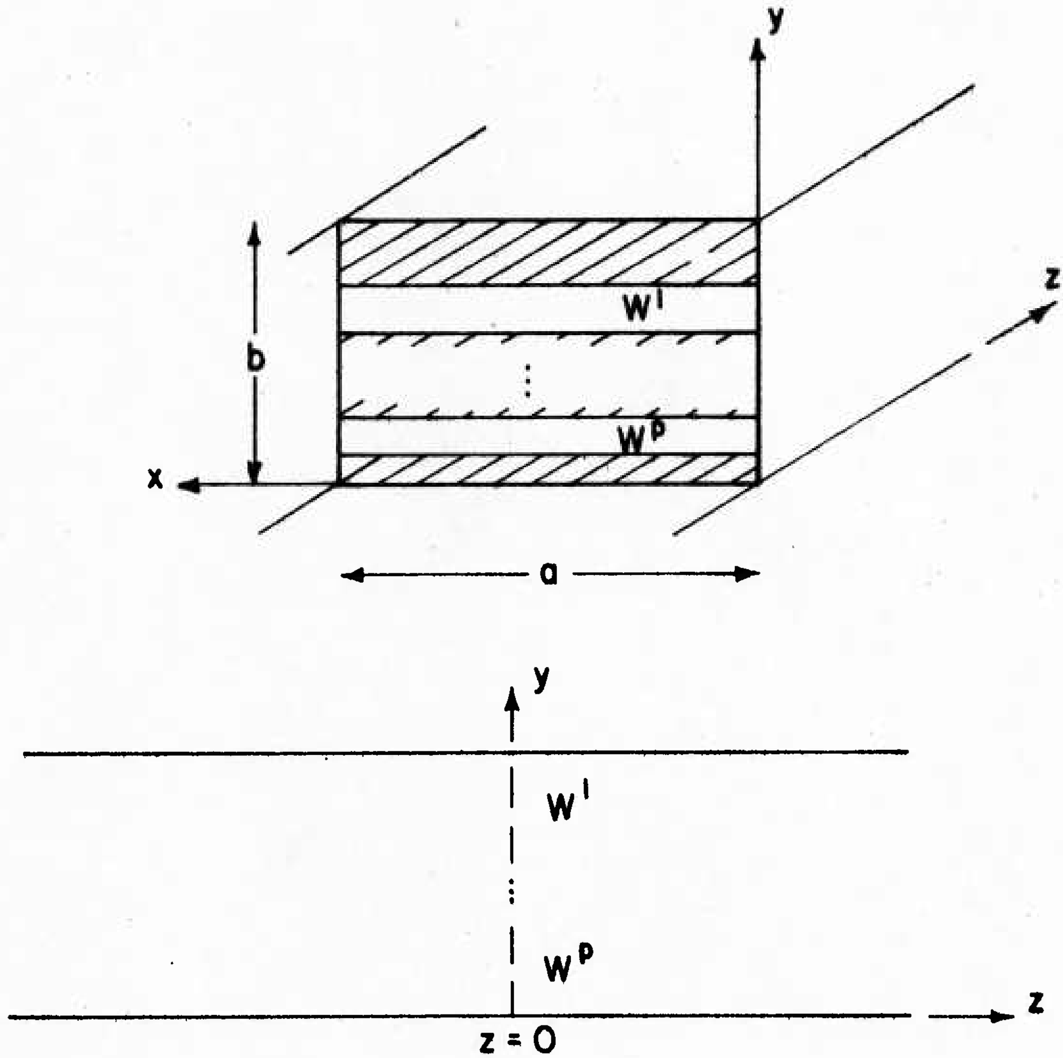

Figure 6. p capacitive windows in a rectangular waveguide.

therefore TE_{1n} to x, since these are the only modes having an H_x component that varies as $\sin(\frac{\pi}{a}x)$ along the x-axis and no E_x component [2, Section 4-4]. The z-transverse field of any such mode has the distribution

$$\left.\begin{aligned} \underline{E}_{n_t} &= e^{\pm\gamma_n z}\,\underline{e}_n \\ \underline{H}_{n_t} &= \mp\,\eta_n\, e^{\pm\gamma_n z}\,\underline{z}\times\underline{e}_n + H_y\,\underline{y} \end{aligned}\right\} \tag{106}$$

where

$$\left.\begin{aligned} \underline{e}_n &= \sqrt{\frac{2\varepsilon_n}{ab}}\,\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\underline{y} \\ \varepsilon_n &= \begin{cases} 1 & n = 0 \\ 2 & n \geq 1 \end{cases} \end{aligned}\right\} \tag{107}$$

and

$$\begin{aligned} \eta_n &= \frac{\gamma_0}{\gamma_n}\,\eta_0 \\ \gamma_n &= \sqrt{\left(\frac{n\pi}{b}\right)^2 + \left(\frac{\pi}{a}\right)^2 - \kappa^2}\ . \end{aligned} \tag{108}$$

The modal expansions in Sections 1, 2, and 4 are in terms of TE and TM to z modes. However, any other complete set of modes can be utilized. Only, then, the appropriate modal characteristic admittances have to be used in (3), while everything else remains unchanged. Thus, for the problem of capacitive windows, a set of TE_{1n} to x modes is clearly the natural choice, and (2) and (3) are to be replaced by (108).

The total z-transverse field has only a y-component of electric field that varies as $\sin(\frac{\pi}{a}x)$ along the x-axis. It then follows from (5) that each $\underline{M}^m$ has only an x-component that varies as $\sin(\frac{\pi}{a}x)$:

$$\underline{M}^m = \sin(\frac{\pi}{a}x)\, M^m(y)\, \underline{x} \qquad \text{on} \quad W^m . \tag{109}$$

The H_y component in (106) does not therefore figure in the analysis.

The generalized network representation of the window is given by (96) and depicted in Figure 5. Here, however, $\bar{\bar{Y}}$ is the p by p block matrix whose ℓmth block is the matrix

$$\begin{aligned} \bar{\bar{Y}}^{\ell m} &= [\bar{\bar{Y}}^{\ell m}_{ij}]_{p^\ell \times p^m} \\ &= [\sum_{n=0}^{\infty} \eta_n \int_{W^\ell} \sin(\frac{\pi}{a}x')\, M_i^\ell\, e_n ds' \int_{W^m} e_n \sin(\frac{\pi}{a}x)\, M_j^m ds]_{p^\ell \times p^m} \end{aligned} \tag{110}$$

and $\vec{I}$ is the p segment vector whose ℓth segment is the vector

$$\vec{I}^\ell = [I_i^\ell]_{p^\ell \times 1} = [-\eta_0 \int_{W^\ell} \sin(\frac{\pi}{a}x)\, M_i\, e_0 ds]_{p^\ell \times 1} \tag{111}$$

as can readily be verified by following steps similar to those in Section 6.

The scattering matrix is analogously given by

$$S = \begin{bmatrix} a_0 & d_0' \\ & \\ b_0 & a_0' \end{bmatrix} \tag{112}$$

where

$$a_0 = a_0' = 1 - \sum_{m=1}^{p} \sum_{j=1}^{p^m} V_j^m \int_{W^m} e_0 \sin\left(\frac{\pi}{a} x\right) M_j^m \, ds \tag{113}$$

$$b_0 = d_0' = 1 + a_0 = 1 + a_0'. \tag{114}$$

Like that of the inductive windows, the scattering matrix of the capacitive windows is both symmetric and unitary. The second term in the left-hand side of (82) now account for the reactive power due to TE_{1n} to x, $n > 1$, modes.

8. The Impedance Matrix of the Windows

Let TE_{10} to z modes of arbitrary amplitudes c_1 and c_2 be incident on the windows (inductive or capacitive) from the left and from the right respectively. Far from the windows, only the same mode can exist.

Let v_1 and v_2 be, respectively, the amplitudes of the E_y component far to the left and to the right of the windows referred to the z=0 plane. It then follows from (1), (43), and (99) for the inductive windows and (112) for the capacitive ones that

$$v_1 = (1 + S_{11})\, c_1 + S_{12}\, c_2 \tag{115}$$

$$v_2 = S_{21}\, c_1 + (1 + S_{22})\, c_2 \tag{116}$$

where S_{ij} ($i,j \in \{1, 2\}$) is the ijth element of the scattering matrix in (99) or (112). The choice of z=0 as a reference plane is only a matter of convenience. In matrix form, (115) and (116) become

$$\vec{v} = (U + S)\, \vec{c}\ . \tag{117}$$

In (117)

$$\vec{v} = \begin{bmatrix} v_1 \\ \\ v_2 \end{bmatrix}$$
$$\vec{c} = \begin{bmatrix} c_1 \\ \\ c_2 \end{bmatrix}\ . \tag{118}$$

Similarly, let i_1 and i_2 be, respectively, the amplitudes of the H_x component far to the left and to the right of the windows extrapolated back to the z=0 plane. Then

$$- \zeta_{10} i_1 = (1 - S_{11})\, c_1 - S_{12}\, c_2 \tag{119}$$

$$\zeta_{10} i_2 = - S_{21}\, c_1 + (1 - S_{22})\, c_2\ . \tag{120}$$

In matrix form, (119) and (120) become

$$\zeta_{10}\, \vec{i} = (U - S)\, \vec{c} \tag{121}$$

where ζ_{10} is the characteristic impedance of the dominant mode, and

$$\vec{i} = \begin{bmatrix} -i_1 \\ \\ i_2 \end{bmatrix}\ . \tag{122}$$

To relate to network theory, let $(v_1, - i_1)$ and (v_2, i_2) be the complex voltage current pairs at the terminals of a two-port network [10, Section 4-5]. Then

$$\vec{v} = Z\,\vec{i} \tag{123}$$

where Z is the network impedance matrix. From (117) and (121), Z is readily found as

$$Z = \zeta_{10}\,(U + S)(U - S)^{-1}. \tag{124}$$

Since S is symmetric, so is Z. Furthermore, since

$$\begin{aligned} Z = \zeta_{10}(U + S)(U - S)^{-1} &= \zeta_{10}(S^H S + S)(S^H S - S)^{-1} \\ &= \zeta_{10}(S^H + U)(S^H - U)^{-1} \\ &= -\,Z^H \end{aligned} \tag{125}$$

the elements of Z are pure imaginary. Finally, using (99), (102), and (103), or (112), (113), and (114), in (124), it becomes

$$\left.\begin{aligned} Z &= j\begin{bmatrix} X & X \\ & \\ X & X \end{bmatrix} \\ jX &= -\ \zeta_{10}\,\frac{1 + S_{11}}{2S_{11}} \end{aligned}\right\} \tag{126}$$

as can easily be verified by carrying out the matrix inversion and multiplication there.

9. The Equivalent Network of the Windows

The effect of the windows on the dominant waveguide mode is described by the windows' impedance matrix Z. Such a representation can be realized in the form of a two-port T-network [10, Section 4-5]. However, because of (126), the equivalent network is merely a shunt reactive element.

The higher order modes excited are evanescent, and are therefore the cause for a localized energy close to the windows. In view of (3)and (88), the energy stored close to the inductive windows is predominantly magnetic, whereas that stored close to the capacitive windows, in view of (108), is predominantly electric. The shunt element in the equivalent network is therefore an inductor for the inductive windows, and a capacitor for the capacitive windows, as can be seen in Figure 7. There

$$jX = \begin{cases} \zeta_{10} \dfrac{1 + a_1}{2a_1} & \text{for inductive windows} \\[2ex] \zeta_{10} \dfrac{1 + a_0}{2a_0} & \text{for capacitive windows.} \end{cases} \tag{127}$$

10. Concluding Remarks

In this chapter, the system of multiple apertures of arbitrary shape in the transverse plane between two cylindrical waveguides has been considered. The analysis is based on the generalized formulation for aperture problems [11].

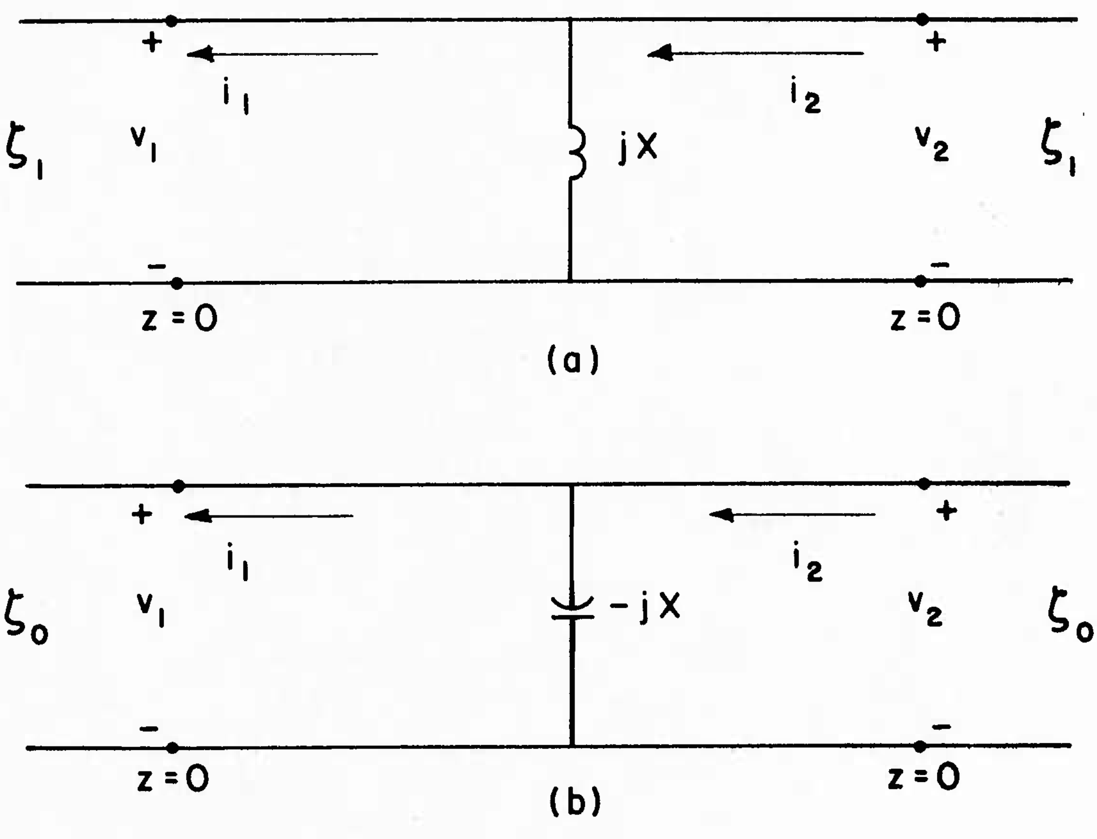

Figure 7. The equivalent network (a) for the inductive windows (b) for the capacitive windows.

For a multi-mode incident field, a representation of the system of apertures in terms of two generalized networks in parallel with current sources is obtained. This representation obeys the two basic network laws: the conservation of complex power law and the reciprocity law. Furthermore, each generalized network depends on the modes of only one waveguide. Thus, for a given collection of apertures, different waveguides can be considered one at a time. The generalized network representation can then be obtained for any required combination.

The scattering matrix of the apertures is then deduced from the generalized network representation. Although the aperture-waveguide structure is both reciprocal and lossless, the scattering matrix is neither symmetric nor unitary. This is because of the different characteristic admittances of the modes, and the consideration of evanescent modes. The scattering matrix can be made symmetric by using a different mode normalization from (4), but no such normalization can make the scattering matrix unitary if evanescent modes are present. A detailed discussion of this point is given in [12]. Since the scattering submatrices are expressed in terms of the generalized networks, given a set of apertures, the scattering matrix for a combination of two waveguides can be obtained by combining the generalized network for one waveguide with that of the other.

Inductive and capacitive windows in a rectangular waveguide are then considered as special cases of the general problem. The windows scattering matrix, since it involves only the dominant waveguide mode, a propagating mode, is both symmetric and unitary. The windows' impedance matrix is then obtained, and readily realized as a shunt reactive element. Other problems of interest that can be worked out as special cases are those of inductive and capacitive windows in a rectangular waveguide loaded with different dielectrics on both sides of the windows, and of coupling through small apertures. The latter is discussed in a more general setting using related methods in [13].

The analysis in this chapter is basically theoretical, and is presented so that all the results and different relationships are clearly seen. Results that relate the present procedure to the mode matching and conservation of complex power techniques are given in [14].

Chapter 3

MULTIPLE INDUCTIVE POSTS IN A RECTANGULAR WAVEGUIDE

Consider a system of posts $P^1, P^2, \ldots, P^P$ located close to each other in a rectangular waveguide. These posts are assumed perfectly conducting, of arbitrary shape and thickness, and uniform along the narrow side of the waveguide, i.e., of the inductive type. The medium filling the waveguide is assumed linear, homogeneous, isotropic, and dissipation free, and is therefore characterized by the real scalar permittivity ε and the real scalar permeability μ. The problem considered is depicted in Figure 1.

1. Preliminary Considerations

Let a TE_{10} to z mode of unit amplitude be incident on the posts from the left. This mode has the field distribution

$$\left.\begin{aligned} E_y^i &= \sin\left(\frac{\pi}{a}x\right) e^{-\gamma_1 z} \\ H_x^i &= \frac{-\gamma_1}{j\omega\mu} \sin\left(\frac{\pi}{a}x\right) e^{-\gamma_1 z} \\ H_z^i &= \frac{-\pi}{j\omega\mu a} \cos\left(\frac{\pi}{a}x\right) e^{-\gamma_1 z} \end{aligned}\right\} \tag{1}$$

where

$$\left.\begin{aligned} \gamma_1 &= j\,\frac{2\pi}{\lambda_1} = j\sqrt{\kappa^2 - \left(\frac{\pi}{a}\right)^2} \\ \kappa &= \frac{2\pi}{\lambda} = \omega\sqrt{\mu\varepsilon}\,. \end{aligned}\right\} \tag{2}$$

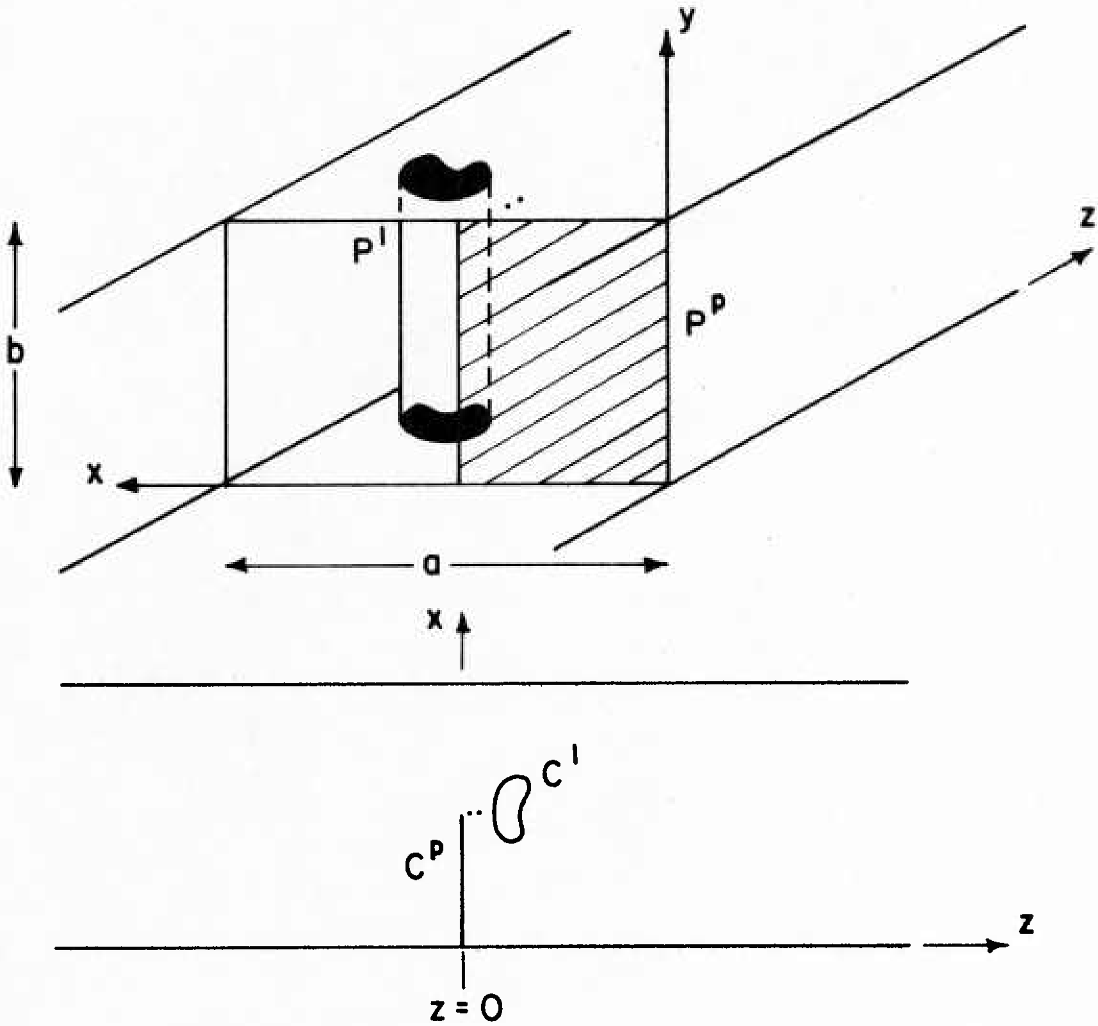

Figure 1. p inductive posts in a rectangular waveguide.

In (2), κ is the wave number of the waveguide medium, and λ is its wave length. Furthermore, it is assumed that $a < \lambda < 2a$ and $2b < \lambda$ so that only the dominant mode can propagate in the waveguide.

Since each post is uniform along the y-axis, and since the exciting mode has no y-component of magnetic field, neither does the scattered field. That is, the scattered field is TM to y, and can therefore be derived from a magnetic vector potential $\underline{A}$ having only a y-component ϕ [2, Section 8-7]:

$$A = \phi \underline{y} \quad . \tag{3}$$

The scattered field is given in terms of ϕ by

$$\left.\begin{aligned} \underline{E}^s &= \frac{1}{j\omega\varepsilon} \nabla \times \nabla \times \phi\, \underline{y} \\ \underline{H}^s &= \nabla \times \phi \underline{y} \end{aligned}\right\} \tag{4}$$

while ϕ itself satisfies

$$(\nabla^2 + \kappa^2)\ \phi = 0 \ . \tag{5}$$

Expanding (4) in rectangular coordinates, the components of the scattered field are found to be

$$\left.\begin{aligned} E_x^s &= \frac{1}{j\omega\varepsilon} \frac{\partial^2}{\partial y \partial x}\ \phi \\ E_y^s &= \frac{1}{j\omega\varepsilon} \left(\frac{\partial^2}{\partial y^2} + \kappa^2\right)\phi \\ E_z^s &= \frac{1}{j\omega\varepsilon} \frac{\partial^2}{\partial y \partial z}\ \phi \\ H_x^s &= -\frac{\partial}{\partial z}\ \phi \\ H_y^s &= 0 \\ H_z^s &= \frac{\partial}{\partial x}\ \phi. \end{aligned}\right\} \tag{6}$$

Furthermore, since each post is uniform along the y-axis, and since the exciting mode has only a y-component of electric field that does not vary with y, so does the scattered field. It then follows from (6) that ϕ is also independent of y. The only components of the scattered field are now given by

$$\left.\begin{aligned} E_y^s &= -\,j\omega\mu\,\phi\,(x,z) \\ H_x^s &= -\frac{\partial}{\partial z}\,\phi\,(x,z) \\ H_z^s &= \frac{\partial}{\partial x}\,\phi\,(x,z). \end{aligned}\right\} \qquad (7)$$

The total field, incident plus scattered, must have zero tangential electric field at the waveguide walls. The incident field is a free waveguide mode, and does therefore have zero electric field tangent to the walls. The scattered field must then have zero tangential electric field at the walls. This is readily accomplished by setting

$$\phi(x,z) = 0, \quad x = 0,\ a, \text{ and all } y \text{ and } z. \qquad (8)$$

The boundary conditions (8), once satisfied for any value of y, are clearly satisfied for all values of y. Thus, the problem is basically a two-dimensional scalar one that can entirely be worked out in some y=constant plane within the waveguide.

In the next section, the Green's function for the TM_{n0} to y modes in a rectangular waveguide is obtained. This is then used to determined ϕ.

2. The Green's Function for TM_{n0} to y Modes in a Rectangular Waveguide

Consider a uniform electric current filament $\underline{J}$ directed across the waveguide parallel to the y-axis and located at (x', z') as shown in Figure 2.

Since $\underline{J}$ is directed along the y-axis, the field produced must have only a y-component of electric field and no y-component of magnetic field. Furthermore, since $\underline{J}$ is uniform along the y-axis, so must be E_y. This can readily be established, for instance, by the reciprocity theorem [2, Section 3-8]. A magnetic vector potential having only a y-component ϕ, proportional to E_y, as is seen in Section 1, can then be used to derive all field components.

Only TM_{n0} to y (TE_{n0} to z) modes can be excited in the waveguide, since these are the only modes having only an E_y component that does not vary with y and no H_y component [2, Sections 3-4,4]. The potential function ϕ due to the filament, relabeled G, is therefore referred to as the Green's function for TM_{n0} to y (TE_{n0} to z) modes in a rectangular waveguide. Below, G is found as a series of these modes.

The wave equation satisfied by G, for each y, is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \kappa^2\right) G(x,z) = - \delta(x-x')\, \delta(z-z') \tag{9}$$

subject to the boundary conditions

$$G(x,z) = 0, \quad x = 0,a, \quad \text{and all } z. \tag{10}$$

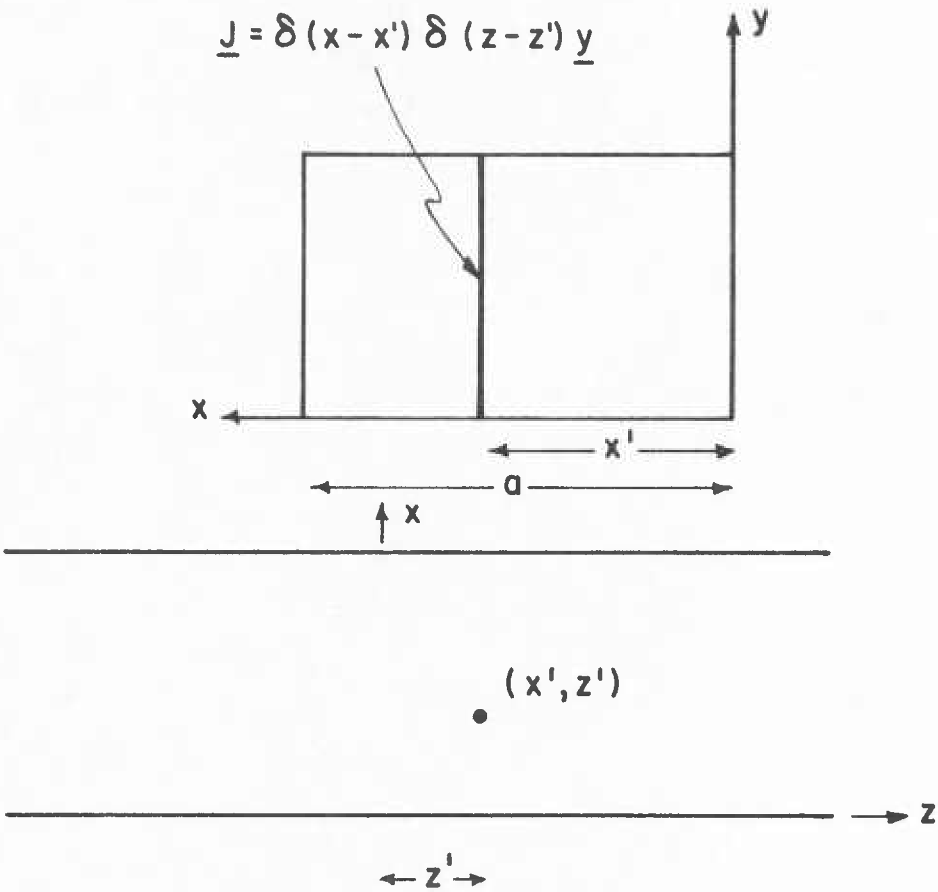

Figure 2. An electric current filament $\underline{J}$ in a recgangular waveguide parallel to the y-axis.

In (10), δ is the Dirac delta function. Multiplying throughout (9) by $\sin\left(\frac{n\pi}{a}x\right)$, then integrating over x from 0 to a, it becomes

$$\left(\frac{d^2}{dz^2} + \kappa^2 - \left(\frac{n\pi}{a}\right)^2\right) \int_0^a G(x,z) \sin\left(\frac{n\pi}{a}x\right) dx$$

$$= - \sin\left(\frac{n\pi}{a}x'\right) \delta(z-z') . \tag{11}$$

Put

$$G_n(z) = \int_0^a G(x,z) \sin\left(\frac{n\pi}{a}x\right) dx$$

$$n = 1,2,\ldots . \tag{12}$$

$$\gamma_n = \sqrt{\left(\frac{n\pi}{a}\right)^2 - \kappa^2}$$

The one-dimensional wave equation (11) then becomes

$$\left(\frac{d^2}{dz^2} - \gamma_n^2\right) G_n(z) = - \sin\left(\frac{n\pi}{a}x'\right) \delta(z-z') . \tag{13}$$

For the solution of (13) to represent waves traveling away from the filament, G_n must be of the form

$$G_n(z) = \begin{cases} A_n e^{-\gamma_n z} & z > z' \\ B_n e^{\gamma_n z} & z < z' \end{cases} \tag{14}$$

where A_n and B_n are constants to be determined. Since G is proportional to E_y, it is continuous across the filament at z=z' [2, Section 1-14], and so is G_n. Thus

$$A_n e^{-\gamma_n z'} - B_n e^{\gamma_n z'} = 0 . \tag{15}$$

Furthermore, integrating (13) over z from $z'-\Delta$ to $z'+\Delta$, then letting Δ go to zero, it becomes

$$\frac{d}{dz} G_n(z)\Bigg|_{z'_-}^{z'_+} = -\sin\left(\frac{n\pi}{a}x'\right) . \tag{16}$$

That is, $\frac{d}{dz} G_n$ is discontinuous at $z = z'$ by the amount $-\sin\left(\frac{n\pi}{a}x'\right)$. Thus

$$A_n\gamma_n e^{-\gamma_n z'} + B_n\gamma_n e^{\gamma_n z'} = \sin\left(\frac{n\pi}{a}x'\right) . \tag{17}$$

Solving (15) and (17) simultaneously, A_n and B_n are found to be

$$A_n = \frac{1}{2\gamma_n} \sin\left(\frac{n\pi}{a}x'\right) e^{\gamma_n z'} \tag{18}$$

$$B_n = \frac{1}{2\gamma_n} \sin\left(\frac{n\pi}{a}x'\right) e^{-\gamma_n z'} . \tag{19}$$

Combining (18) and (19) with (14), G_n becones

$$G_n(z) = \frac{1}{2\gamma_n} \sin\left(\frac{n\pi}{a}x'\right) e^{-\gamma_n|z-z'|} , \quad n=1,2,\dots . \tag{20}$$

By Fourier theory [15, Section 43] , (12) can be inverted as

$$G(x,z|x',z') = \frac{1}{a}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x'\right) e^{-\gamma_n|z-z'|}}{\gamma_n} . \tag{21}$$

Clearly, G satisfies the boundary conditions (10).

3. Basic Formulation

Let $(\underline{E}^i, \underline{H}^i)$ be incident while all the posts are absent, and $(\underline{E}(\underline{J}), \underline{H}(\underline{J}))$ be the field produced by an electric current of density $\underline{J} = \bigcup_{m=1}^{p} \underline{J}^m$, where $\underline{J}^m$ is the current on P^m, while all the posts are absent. By the uniqueness theorem [2, Section 3-3], $(\underline{E}^i + \underline{E}(\underline{J}), \underline{H}^i + \underline{H}(\underline{J}))$ is identical with the original field whenever

$$\underline{n}^m \times (\underline{E}^i + \underline{E}(\underline{J})) = 0 \quad \text{on } P^m . \tag{22}$$

In (22), $\underline{n}^m$ is the outward unit vector normal to P^m. $(\underline{E}(\underline{J}), \underline{H}(\underline{J}))$ must then have the field distribution (7). Since

$$\underline{n}^m \times \left(\underline{H}(\underline{J})\Big|_{\nu=0_+} - \underline{H}(\underline{J})\Big|_{\nu=0_-}\right) = \underline{J}^m \quad \text{on } P^m \tag{23}$$

where ν is the distance along $\underline{n}^m$ from P^m, $\underline{J}^m$ has only a y-component that does not vary with y:

$$\underline{J}^m = J^m(x,z)\ \underline{y} \quad . \tag{24}$$

As is pointed out in Section 1, the problem is a two-dimensional scalar one that can be worked out in some y=constant plane within the waveguide. Thus, all source and field points are, hereafter, assumed located in any such plane.

By definition, $G(x,z|x',z')\underline{y}$ is the magnetic vector potential produced at any point (x,z) by a unit electric current filament in the y-direction located at (x',z'). By superposition, then,

$\underline{J}$ produces at (x,z) the magnetic vector potential $\phi\, \underline{y}$, where

$$\left.\begin{aligned} \phi(x,z) &= \sum_{m=1}^{p} \int_{C^m} J^m(x',z')\, G(x,z|x',z')\, dt' \\ dt' &= \sqrt{(dx')^2 + (dz')^2}\quad . \end{aligned}\right\} \tag{25}$$

In (25), G is given by (21), and primed and unprimed coordinates denote, respectively, source and field points.

Since $G(x,z|x',z')$ is a solution of the homogeneous wave equation (5) for all $(x,z) \neq (x',z')$, so is ϕ. Furthermore, ϕ satisfies the boundary conditions (8) by virtue of (10). Thus ϕ, and consequently the complete field solution, can be found once all $\underline{J}^m$ are known. Using (1), (7), and (25), (22) becomes

$$\sin\left(\frac{\pi}{a}\, x\right) e^{-\gamma_1 z} - j\omega\mu \sum_{m=1}^{p} \int_{C^m} J^m(x',z')\, G(x,z|x',z'\, dt' = 0,$$

$$(x,z) \in C^{\ell}, \quad 1 \leq \ell \leq p \tag{26}$$

which is an integral equation for $\underline{J}$.

The higher order (n > 1) modes are evanescent , i.e., decay exponentially with distance from the posts. Thus, at sufficiently large distances, only the dominant (n = 1) mode can exist in the waveguide. The reflection coefficient of the dominant mode is readily found from (7), (21), and (25) as

$$\Gamma = -\frac{j\omega\mu}{a\gamma_1} \sum_{m=1}^{p} \int_{C^m} J^m(x',z') \sin\left(\frac{\pi}{a}\, x'\right) e^{-\gamma_1 z'}\, dt' \ . \tag{27}$$

The transmission coefficient of the dominant mode is then

$$T = 1 - \frac{j\omega\mu}{a\gamma_1} \sum_{m=1}^{P} \int_{C^m} J^m(x',z') \sin\left(\frac{\pi}{a} x'\right) e^{\gamma_1 z'} \, dt' . \tag{28}$$

4. The Scattering Matrix

Following Montgomery et al. [10, Section 5-14], the scattering matrix of the posts is defined as

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} . \tag{29}$$

In (29), S_{11} and S_{21} are, respectively, the amplitudes of the dominant mode reflected to the left and transmitted to the right of the posts due to an incident TE_{10} to z mode of unit amplitude from the left. Consequently, S_{11} and S_{21} are given by (27) and (28), respectively.

Similarly, S_{22} and S_{12} are, respectively, the reflection and transmission coefficients of a TE_{10} to z mode of unit amplitude incident on the posts from the right. This mode has the field distribution

$$\left.\begin{aligned} E_y^i &= \sin\left(\frac{\pi}{a} x\right) e^{\gamma_1 z} \\ E_x^i &= \frac{\gamma_1}{j\omega\mu} \sin\left(\frac{\pi}{a} x\right) e^{\gamma_1 z} \\ H_z^i &= \frac{-\pi}{j\omega\mu a} \cos\left(\frac{\pi}{a} x\right) e^{\gamma_1 z} . \end{aligned}\right\} \tag{30}$$

The previous analysis carries through in this case. Thus, the scattered field is given by (7) and (25), but with $\underline{J}' = \bigcup_{m=1}^{p} \underline{J}'^{m}$, now replacing $\underline{J}$ in (25), determined by solving the integral equation

$$\sin\left(\frac{\pi}{a}x\right) e^{\gamma_1 z} - j\omega\mu \sum_{m=1}^{p} \int_{C^m} J'^{m}(x',z')\, G(x,z|x',z')\, dt' = 0,$$

$$(x,z) \in C^{\ell}, \qquad 1 \leq \ell \leq p \tag{31}$$

rather than (26). It then follows from (7), (21), and (25) that

$$S_{22} = -\frac{j\omega\mu}{a\gamma_1} \sum_{m=1}^{p} \int_{C^m} J'^{m}(x',z') \sin\left(\frac{\pi}{a}x'\right) e^{\gamma_1 z'}\, dt' \tag{32}$$

$$S_{12} = 1 - \frac{j\omega\mu}{a\gamma_1} \sum_{m=1}^{p} \int_{C^m} J'^{m}(x',z') \sin\left(\frac{\pi}{a}x'\right) e^{-\gamma_1 z'}\, dt' . \tag{33}$$

The scattering matrix is both symmetric and unitary. That is

$$\left.\begin{aligned} S &= S^T \\ SS^H &= S^H S = U \end{aligned}\right\} \tag{34}$$

where T and H denote, respectively, matrix transpose and Hermitian, and U is the identity matrix.

Let $(\underline{E}^1, \underline{H}^1)$ and $(\underline{E}^2, \underline{H}^2)$ be the z-transverse fields in the waveguide, sufficiently far from the posts, due to TE_{10} to z modes of arbitrary amplitudes c_1 and c_2 incident from the left and from the right of the posts, respectively. It then follows from (1), (7), (21), (25), (27), (28), (30), (32), and (33) that

$$\underline{E}^1 = \begin{cases} c_1(e^{-\gamma_1 z} + S_{11}\, e^{\gamma_1 z}) \sin(\frac{\pi}{a} x)\, \underline{y} & z \ll 0 \\ c_1 S_{21} \sin(\frac{\pi}{a} x)\, e^{-\gamma_1 z}\, \underline{y} & z \gg 0 \end{cases}$$

$$\underline{H}^1 = \begin{cases} -\eta_1 c_1(e^{-\gamma_1 z} - S_{11}\, e^{\gamma_1 z}) \sin(\frac{\pi}{a} x)\, \underline{x} & z \ll 0 \\ -\eta_1 c_1 S_{21} \sin(\frac{\pi}{a} x)\, e^{-\gamma_1 z}\, \underline{x} & z \gg 0 \end{cases} \tag{35}$$

$$\underline{E}^2 = \begin{cases} c_2 S_{12} \sin(\frac{\pi}{a} x)\, e^{\gamma_1 z}\, \underline{y} & z \ll 0 \\ c_2(e^{\gamma_1 z} + S_{22}\, e^{-\gamma_1 z}) \sin(\frac{\pi}{a} x)\, \underline{y} & z \gg 0 \end{cases}$$

$$\underline{H}^2 = \begin{cases} \eta_1 c_2 S_{12} \sin(\frac{\pi}{a} x)\, e^{\gamma_1 z}\, \underline{x} & z \ll 0 \\ \eta_1 c_2(e^{\gamma_1 z} - S_{22}\, e^{-\gamma_1 z}) \sin(\frac{\pi}{a} x)\, \underline{x} & z \gg 0. \end{cases}$$

In (35), η_1 is the characteristic admittance of the dominant waveguide mode:

$$\eta_1 = \frac{1}{\zeta_1} = \frac{\gamma_1}{j\omega\mu} \quad . \tag{36}$$

Let W be the closed surface consisting of all metallic walls between the two waveguide cross sections at $z = z_1$ and z_2, for some $z_1 \ll 0$ and $z_2 \gg 0$, and these two cross sections. The reciprocity theorem then states that

$$\int_W (\underline{E}^1 \times \underline{H}^2 - \underline{E}^2 \times \underline{H}^1) \cdot \underline{n}\, ds = 0 \tag{37}$$

where $\underline{n}$ is the outward unit vector normal to W. Substituting (35) into (37), there then results

$$ab\eta_1 c_1 S_{12} c_2 = ab\eta_1 c_1 S_{21} c_2 \tag{38}$$

whence

$$S_{12} = S_{21} \ . \tag{39}$$

The scattering matrix is symmetric if and only if the whole structure is reciprocal.

That S is unitary follows from conservation of power considerations. Let the two dominant modes be simultaneously incident on the posts from the left and from the right. The complex power scattered far to the left and to the right of the posts is basically

$$P_{sc} = \frac{ab}{2}\eta_1(|c_1 S_{11} + c_2 S_{12}|^2 + |c_1 S_{21} + c_2 S_{22}|^2) \tag{40}$$

whereas that incident is given by

$$P_{in} = \frac{ab}{2}\eta_1(|c_1|^2 + |c_2|^2) \ . \tag{41}$$

Since the structure is lossless, and since P_{in} and P_{sc} are real, they must be equal. Put

$$\vec{c} = \begin{bmatrix} c_1 \\ \\ c_2 \end{bmatrix} \ . \tag{42}$$

Then

$$\frac{ab}{2}\eta_1 \vec{c}^H\vec{c} = \frac{ab}{2}\eta_1 \vec{c}^H S^H S \vec{c} \tag{43}$$

or

$$S^H S = U. \tag{44}$$

5. The Impedance Matrix

Let TE_{10} to z modes of arbitrary amplitudes c_1 and c_2 be incident on the posts from the left and from the right, respectively.

Let v_1 and v_2 be, respectively, the amplitudes of the E_y component far to the left and to the right of the posts referred to the $z = 0$ plane. It then follows from (35) that

$$v_1 = (1 + S_{11})\, c_1 + S_{12}\, c_2 \tag{45}$$

$$v_2 = S_{21}\, c_1 + (1 + S_{22})\, c_2\ . \tag{46}$$

The choice of $z = 0$ as a reference plane is only a matter of convenience. In matrix form, (45) and (46) become

$$\vec{v} = (U + S)\, \vec{c} \tag{47}$$

where

$$\vec{v} = \begin{bmatrix} v_1 \\ \\ v_2 \end{bmatrix}\ . \tag{48}$$

Similarly, let i_1 and i_2 be, respectively, the amplitudes of the H_x component far to the left and to the right of the posts extrapolated back to the $z = 0$ plane. Then

$$-\zeta_1 i_1 = (1 - S_{11})\, c_1 - S_{12} c_2 \tag{49}$$

$$\zeta_1 i_2 = -\ S_{21} c_1 + (1 - S_{22})\, c_2\ . \tag{50}$$

In matrix form, (49) and (50) become

$$\zeta_1\, \vec{i} = (U - S)\, \vec{c} \tag{51}$$

where

$$i = \begin{bmatrix} -i_1 \\ \\ i_2 \end{bmatrix} . \tag{52}$$

To relate to network theory, let $(v_1, - i_1)$ and (v_2, i_2) be the complex voltage-current pairs at the terminals of a two-part network [10, Section 5-2]. Then

$$\vec{v} = Z \vec{i} \tag{53}$$

where Z is the network impedance matrix. From (47) and (51), Z is readily found as

$$Z = \zeta_1 (U + S)(U - S)^{-1} . \tag{54}$$

Since S is symmetric, so is Z. Furthermore, since

$$\begin{aligned} Z = \zeta_1 (U + S)(U - S)^{-1} &= \zeta_1 \, (S^H S + S)(S^H S - S)^{-1} \\ &= \zeta_1 \, (S^H + U)(S^H - U)^{-1} \\ &= - Z^H \end{aligned} \tag{55}$$

the elements of Z are pure imaginary. Thus

$$\left.\begin{aligned} Z &= j \begin{bmatrix} X_{11} & X_{12} \\ \\ X_{21} & X_{22} \end{bmatrix} \\ X_{12} &= X_{21} . \end{aligned}\right\} \tag{56}$$

6. The Equivalent Network

The complete field solution is seldom needed. Rather, the effect of the posts on the dominant waveguide mode is what must accurately be described. From an engineering perspective, a description in terms of a network of lumped elements is preferred.

The effect of the posts on the dominant waveguide mode is fully described by the posts' impedance matrix Z. Such a representation can be realized in the form of a two-port T-network [10, Section 4-5].

The characteristic impedances of the TM_{n0} to y modes are given by

$$\left.\begin{aligned} \zeta_n &= j \frac{\zeta\kappa}{\sqrt{(\frac{n\pi}{a})^2 - \kappa^2}} , \qquad n > 1 \\ \zeta &= \sqrt{\frac{\mu}{\varepsilon}} . \end{aligned}\right\} \tag{57}$$

Since these modes are evanescent, the energy stored close to the posts, in view of (57), is predominantly magnetic. This effect can suitably be represented by an inductor in the shunt arm of the network. The elements in the series arms, however, are capacitors to account for the charge difference across the posts in the z direction. The equivalent network of the posts is shown in Figure 3.

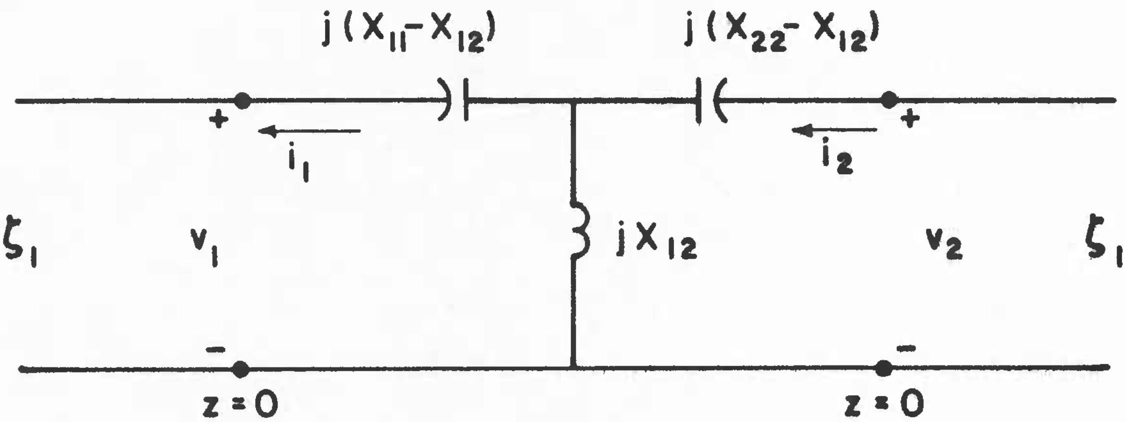

Figure 3. The equivalent network of the posts.

7. Solution of the Integral Equation

The integral equation (26) can be put in the compact form

$$\left.\begin{aligned} &\sum_{m=1}^{p} Z^m(J^m) = V \\ &Z^m(J^m) = j\omega\mu \int_{C^m} J^m(x',z')\, G(x,z|x',z')\, dt' \\ &V = \sin\left(\frac{\pi}{a} x\right) e^{-\gamma_1 z}, \qquad (x,z) \in C^\ell,\ 1 \leq \ell \leq p\ . \end{aligned}\right\} \tag{58}$$

An exact solution of (58) can rarely be obtained, and an approximate solution has then to be sought.

Let each C^m be approximated by a polygon $\sum^m = \{S_1^m, S_2^m, \ldots, S_{q^m}^m\}$ as shown in Figure 4, and put

$$J^m(x',z') \simeq \sum_{j=1}^{q^m} I_j^m\, J_j^m(x',z')\ . \tag{59}$$

In (59), I_j^m are complex coefficients to be determined, whereas each J_j^m is a real function that vanishes on all $S_{i \neq j}^{\ell \neq m}$, but is otherwise unspecified. Substituting (59) into (58), it becomes

$$\sum_{m=1}^{p} \sum_{j=1}^{q^m} I_j^m\, Z_j^m(J_j^m) + r = V\ ,$$

$$(x,z) \in S_i^\ell,\quad 1 \leq \ell \leq p,\quad 1 \leq i \leq q^\ell \tag{60}$$

where $Z_j^m(J_j^m)$ is given by (48) except that the integration is taken over S_j^m, and r is a residual term. A Galerkin solution [4, Section 1-3] can be obtained by requiring that r be orthogonal to all J_j^m.

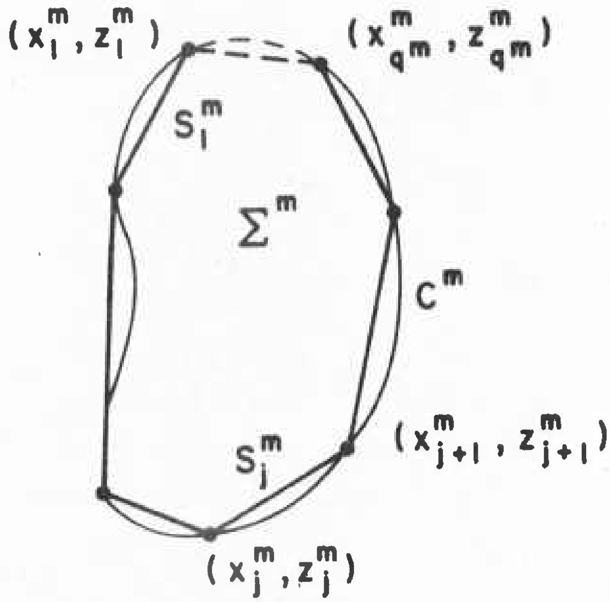

Figure 4. C^m approximated by a polygon $\sum^m$.

Define the inner product

$$\langle A, B\rangle = \int_{\bigcup_{\ell=1}^{p} \Sigma^{\ell}} A\, B^{*}\, dt \quad . \tag{61}$$

Taking the inner product of (60) with each J_i^{ℓ}, and enforcing the Galerkin condition

$$\langle r, J_i^{\ell}\rangle = 0 \ , \qquad 1 \le \ell \le p \ , \quad 1 \le i \le q^{\ell} \tag{62}$$

there then results the system of equations

$$\sum_{m=1}^{p} \sum_{j=1}^{q^m} I_j^m \langle Z_j^m(J_j^m),\ J_i^{\ell}\rangle = \langle V,\ J_i^{\ell}\rangle \ ,$$

$$1 \le \ell \le p \ , \quad 1 \le i \le q^{\ell} \ . \tag{63}$$

In matrix form, (63) becomes

$$\bar{\bar{Z}}\ \vec{I} = \vec{V} \tag{64}$$

where $\bar{\bar{Z}}$ is a p by p block matrix whose ℓmth block is the matrix

$$Z^{\ell m} = [Z_{ij}^{\ell m}]_{q^{\ell}\times q^{m}} = [\langle Z_j^m(J_j^m),\ J_i^{\ell}\rangle]_{q^{\ell}\times q^{m}} \tag{65}$$

and $\vec{I}$ and $\vec{V}$ are p segment vectors whose mth and ℓth segments are the vectors

$$\vec{I}^{m} = [I_j^m]_{q^m\times 1} \tag{66}$$

$$\vec{V}^{\ell} = [V_i^{\ell}]_{q^{\ell}\times 1} = [\langle V,\ J_i^{\ell}\rangle]_{q^{\ell}\times 1} \tag{67}$$

respectively.

The currents J^m given by (59), with the coefficients I_j^m determined from (65), form the Galerkin solution of (58). A Galerkin solution of (31) can be obtained in a similar manner. Clearly, then, using the same J_j^m, the solution is given by (59), but with the coefficients now determined by solving (64) with the right-hand side vector $\vec{V}$ conjugated.

8. Evaluation of the System of Equations

The construction of $\bar{\bar{Z}}$ in (64) constitutes a large portion of the work involved in the numerical solution. An efficient evaluation of the elements of $\bar{\bar{Z}}$ is therefore necessary for the success of the solution.

The ijth element of the ℓmth block of $\bar{\bar{Z}}$ is given by

$$Z_{ij}^{\ell m} = j\omega\mu \int_{S_i^\ell} J_i^\ell(x,z)\ dt \int_{S_j^m} J_j^m(x',z')\ G(x,z|x',z')\ dt' \qquad (68)$$

where J_j^m are so far unspecified. A particularly simple choice for J_j^m is

$$J_j^m(x',z') = \begin{cases} 1 & (x',z') \in S_j^m \\ \\ 0 & (x',z') \in S_{i \neq j}^{\ell \neq m} \end{cases} \qquad (69)$$

which corresponds to a pulse expansion of J^m. $Z_{ij}^{\ell m}$ then becomes

$$Z_{ij}^{\ell m} = j\omega\mu \int_{S_i^\ell} dt \int_{S_j^m} G(x,z|x',z')\ dt' \ . \qquad (70)$$

Put

$$R_{ij}^{\ell m}(x,z) = \int_{S_j^m} G(x,z|x',z')\, dt', \qquad (x,z) \in S_i^\ell . \tag{71}$$

By the first mean value theorem of integration [16, Section 7-18], there exists a point $(x_o,z_o) \in S_i^\ell$ such that

$$Z_{ij}^{\ell m} = j\omega\mu \quad L_i^\ell\, R_{ij}^{\ell m}(x_o,z_o) \tag{72}$$

where

$$L_i^\ell = \quad (x_{i+1}^\ell - x_i^\ell)^2 + (z_{i+1}^\ell - z_i^\ell)^2 \tag{73}$$

is the length of S_i^ℓ.

The evaluation of $Z_{ij}^{\ell m}$ is now completed by integrating G over S_j^m. Put

$$G(x_o,z_o|x',z') = (G_0 + G_1 + G_2)(x_o,z_o|x',z') . \tag{74}$$

In (74)

$$\left.\begin{aligned}
G_0(x_o,z_o|x',z') &= \frac{1}{j\pi\beta_1} \sin\left(\frac{\pi}{a} x_o\right) \sin\left(\frac{\pi}{a} x'\right) e^{-j\frac{\pi}{a}|z_o-z'|\beta_1} \\
G_1(x_o,z_o|x',z') &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{a} x_o\right) \sin\left(\frac{n\pi}{a} x'\right) e^{-\frac{n\pi}{a}|z_o-z'|} \\
G_2(x_o,z_o|x',z') &= \frac{1}{\pi} \Big[-\sin\left(\frac{\pi}{a} x_o\right) \sin\left(\frac{\pi}{a} x'\right) e^{-\frac{\pi}{a}|z_o-z'|} \\
&\quad + \sum_{n=2}^{\infty} \sin\left(\frac{n\pi}{a} x_o\right) \sin\left(\frac{n\pi}{a} x'\right) \\
&\quad \times \left(\frac{e^{-\frac{\pi}{a}|z_o-z'|\beta_n}}{\beta_n} - \frac{e^{-\frac{n\pi}{a}|z_o-z'|}}{n}\right)\Big]
\end{aligned}\right\} \tag{75}$$

$$\left.\begin{aligned}\gamma_1 &= j\sqrt{\kappa^2 - (\frac{\pi}{a})^2} = j\,\frac{\pi}{a}\sqrt{(\frac{2a}{\lambda})^2 - 1} = j\,\frac{\pi}{a}\,\beta_1 \\ \gamma_n &= \sqrt{(\frac{n\pi}{a})^2 - \kappa^2} = \frac{\pi}{a}\sqrt{n^2 - (\frac{2a}{\lambda})^2} = \frac{\pi}{a}\,\beta_n, \quad n \geq 2 \;.\end{aligned}\right\} \tag{76}$$

The decomposition (74), therefore, amounts to expressing the dynamic Green's function G in terms of a dominant mode wave G_0, the corresponding static Green's function G_1, which can be obtained from (21) by setting κ equal to zero, plus correction terms G_2.

The series defining G_1 is readily summed to give

$$G_1(x_o,z_o|x',z') = \frac{1}{4\pi}\log\left(\frac{\cosh(\frac{\pi}{a}|z_o-z'|) - \cos\frac{\pi}{a}(x_o+x')}{\cosh(\frac{\pi}{a}|z_o-z'|) - \cos\frac{\pi}{a}(x_o-x')}\right) \tag{77}$$

where log denotes the natural logarithm. The details of the summation are given in Appendix A. The series in G_2 is dominated by an exponentially convergent series of positive monotonically decreasing terms (see Appendix B), and can therefore be summed directly at a minimal cost.

The integration of G can be carried out numerically, and for that purpose any quadrature rule can be used. Thus

$$\int_{S_j^m} G(x_o,z_o|x',z')\,dt'$$

$$\simeq \frac{L_j^m}{2}\sum_{k=1}^{N} q_k(G_0 + G_1 + G_2)(x_o,z_o|$$

$$(1 - p_k)x_j^m + p_k x_{j+1}^m,\; (1 - p_k)z_j^m + p_k z_{j+1}^m) \;. \tag{78}$$

In (78), N is the order of the rule, q_k are its coefficients, and p_k determine the location of its abscissas.

When evaluating the diagonal elements of $\bar{\bar{Z}}$, $Z^{\ell\ell}_{ii}$, G_1 offers a logarithmic singularity at $(x_o,z_o) \varepsilon S^{\ell}_i$ that requires particular attention. In Appendix A, the singular part of G_1 is found to be

$$G_{1s}(x_o,z_o|x',z') = -\frac{1}{2\pi} \log \left(\frac{\pi}{a} \sqrt{(x'-x_o)^2 + (z'-z_o)^2}\right). \tag{79}$$

Put

$$G_{1p}(x_o,z_o|x',z') = (G_1 - G_{1s})(x_o,z_o|x',z') . \tag{80}$$

Then

$$\begin{aligned}
&\int_{S^{\ell}_i} G(x_o,z_o|x',z')\, dt' \\
&\quad = \int_{S^{\ell}_i} G_{1s}(x_o,z_o|x',z')\, dt' + \int_{S^{\ell}_i} (G_0+G_{1p}+G_2)(x_o,z_o|x',z')dt' \\
&\quad = -\frac{1}{2\pi} \left[L \log\left(\frac{L}{L^{\ell}_i - L}\right) + L^{\ell}_i(\log \left(\frac{\pi}{a} (L^{\ell}_i - L)\right) - 1)\right] \\
&\qquad + \int_{S^{\ell}_i} (G_0+G_{1p}+G_2)(x_o,z_o|x',z')\, dt' .
\end{aligned} \tag{81}$$

Here, L is the distance between (x_o,z_o) and (x^{ℓ}_i, z^{ℓ}_i). The integral on the right-hand side of (81) has no singularity at (x_o,z_o), and can therefore be evaluated through (78).

The ith element of the ℓth segment of $\vec{V}$ is given by

$$V_i^\ell = \int_{S_i^\ell} J_i^\ell(x,z) \sin\left(\frac{\pi}{a} x\right) e^{-\gamma_1 z} \, dt \tag{82}$$

which, upon using (69), becomes

$$V_i^\ell = \int_{S_i^\ell} \sin\left(\frac{\pi}{a} x\right) e^{-\gamma_1 z} \, dt. \tag{83}$$

The integration in (83) can be carried out exactly. However, a point $(\tilde{x}_o, \tilde{z}_o)$ exists such that

$$V_i^\ell = L_i^\ell \sin\left(\frac{\pi}{a} \tilde{x}_o\right) e^{-\gamma_1 \tilde{z}_o} . \tag{84}$$

Actually, finding such points (x_o, z_o) and $(\tilde{x}_o, \tilde{z}_o)$ is at least as difficult as computing the integrals themselves. For sufficiently small L_i^ℓ, however, the mid-point of S_i^ℓ can replace these points while introducing only very little error. The system of equations thus obtained, clearly, is one that results from enforcing the point matching condition

$$r(x,z) = 0,$$

$$(x,z) \in \left\{ \left(\frac{x_{i+1}^\ell + x_i^\ell}{2}, \frac{z_{i+1}^\ell + z_i^\ell}{2} \right) \middle| \; 1 \le \ell \le p, \; 1 \le i \le q^\ell \right\} \tag{85}$$

in (60) rather than the Galerkin condition (62).

9. Numerical Results

The solution procedure presented is readily translated into a computer program. The elements of the scattering matrix and the reactances of the equivalent T-network are basically the parameters to be computed.

The scattering parameters, thanks to (59) and (69), are computed by

$$\left.\begin{aligned}
S_{11} &\simeq -\frac{j\omega\mu}{a\gamma_1}\sum_{m=1}^{p}\sum_{j=1}^{q^m} I_j^m \int_{S_j^m} \sin\left(\frac{\pi}{a}x'\right) e^{-\gamma_1 '}\, dt' \\
S_{21} &\simeq 1-\frac{j\omega\mu}{a\gamma_1}\sum_{m=1}^{p}\sum_{j=1}^{q^m} I_j^m \int_{S_j^m} \sin\left(\frac{\pi}{a}x'\right) e^{\gamma_1 z'}\, dt' \\
S_{22} &\simeq -\frac{j\omega\mu}{a\gamma_1}\sum_{m=1}^{p}\sum_{j=1}^{q^m} I_j'^m \int_{S_j^m} \sin\left(\frac{\pi}{a}x'\right) e^{\gamma_1 z'}\, dt' \\
S_{12} &\simeq 1-\frac{j\omega\mu}{a\gamma_1}\sum_{m=1}^{p}\sum_{j=1}^{q^m} I_j'^m \int_{S_j^m} \sin\left(\frac{\pi}{a}x'\right) e^{-\gamma_1 z'}\, dt'
\end{aligned}\right\} \tag{86}$$

where I_j^m and $I_j'^m$ are the solutions of (64) with right-hand side vectors $\vec{V}$ and $\vec{V}^*$, respectively. The impedance matrix is then computed through (54). In carrying out the integrations in (86), and also in (78) and (81), an eight-point Gauss-Radau quadrature rule [17] is used. Table 1 shows the p_k and q_k of the rule.

Because of the approximations involved in the solution, however, the scattering matrix need no longer be symmetric nor unitary. To determine the impedance matrix, S_{12} and S_{21} are first replaced by their average

$$S_{av} = \frac{1}{2}(S_{12} + S_{21}) \; . \tag{87}$$

The impedance matrix can then have a non-zero real part.

Table 1. The parameters of an eight-point Gauss-Radau rule.

	1	2	3	4	5	6	7	8
p_k	0.0	0.06412993	0.20414991	0.39535039	$1-p_4$	$1-p_3$	$1-p_2$	$1-p_1$
q_k	0.03571428	0.21070422	0.34112270	0.41245880	q_4	q_3	q_2	q_1

To test the solution, the computer program is run for a few selected problems. In particular, the problems of the circular post, of the symmetrical thin window, and of the triple circular post are considered. Some of the results obtained are plotted in Figures 5-9.

In all the cases, the convergence for the inductive reactance is monotonic and from above, as can readily be seen from Figures 6 and 8. The computed reactances are found to agree well with the data in the Waveguide Handbook (WGHB) [3], with only a few segments needed even for large posts. A complete assessment of the solution performance should also consider the (Frobenius) norm of the real part of Z and the modulus of difference in transmission coefficients. These two numbers are computed in all program runs, and are usually $O(10^{-8})$.

Perhaps the most interesting observation can be drawn by examining Figure 5 for the centered circular post. For large posts $(\frac{d}{a} > 0.25)$, $\frac{X_b}{\zeta_1}\frac{\lambda_1}{2a}$ is no longer frequency independent as is the case with smaller posts $(\frac{d}{a} \leq 0.25)$, but rather branches out. Figure 9 for the symmetrical triple circular post displays yet another almost frequency independent characteristic. This is not surprising, however, since this configuration cancels out the first six higher order modes [18, Section 5-1.3].

10. Concluding Remarks

The system of inductive posts in a rectangular waveguide, i.e., of all the metallic obstacles that are uniform along the narrow side of the waveguide, but are otherwise of arbitrary shape and thickness, has been considered in this chapter.

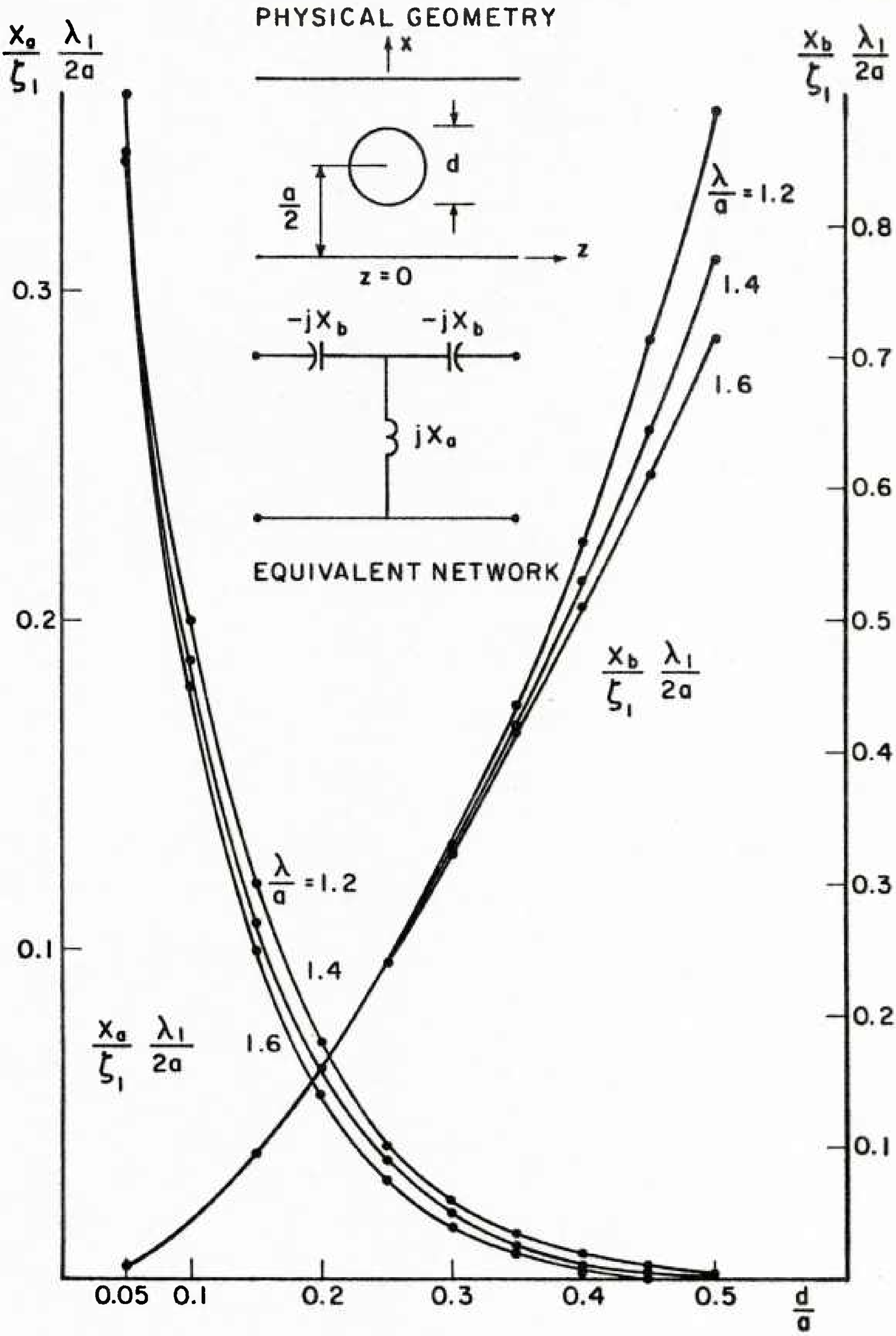

Figure 5. Network reactances of the centered circular post. The number of segments used is 24.

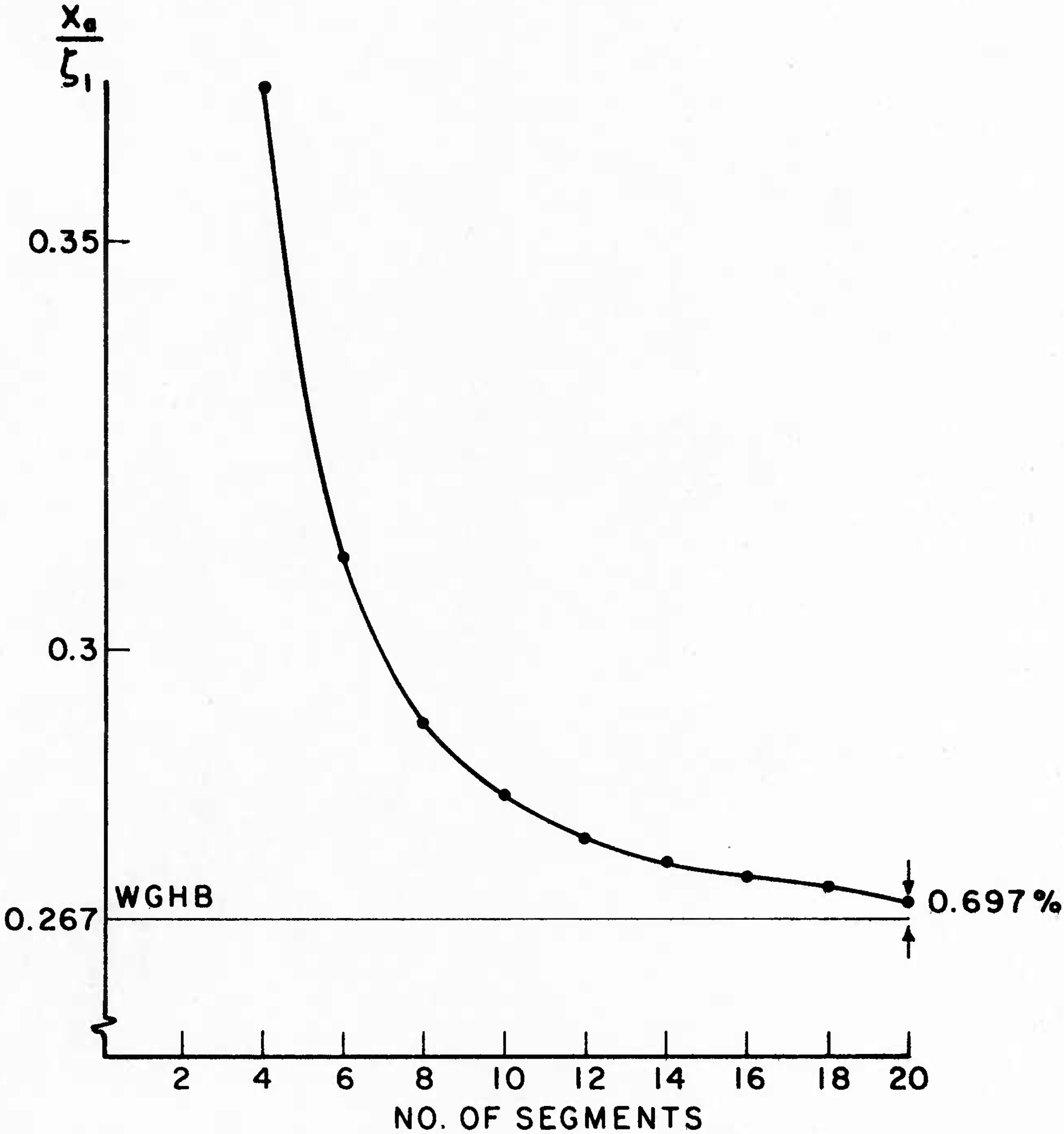

Figure 6. The convergence behavior for the centered circular post ($\frac{\lambda}{a}$ = 1.2, $\frac{d}{a}$ = 0.1).

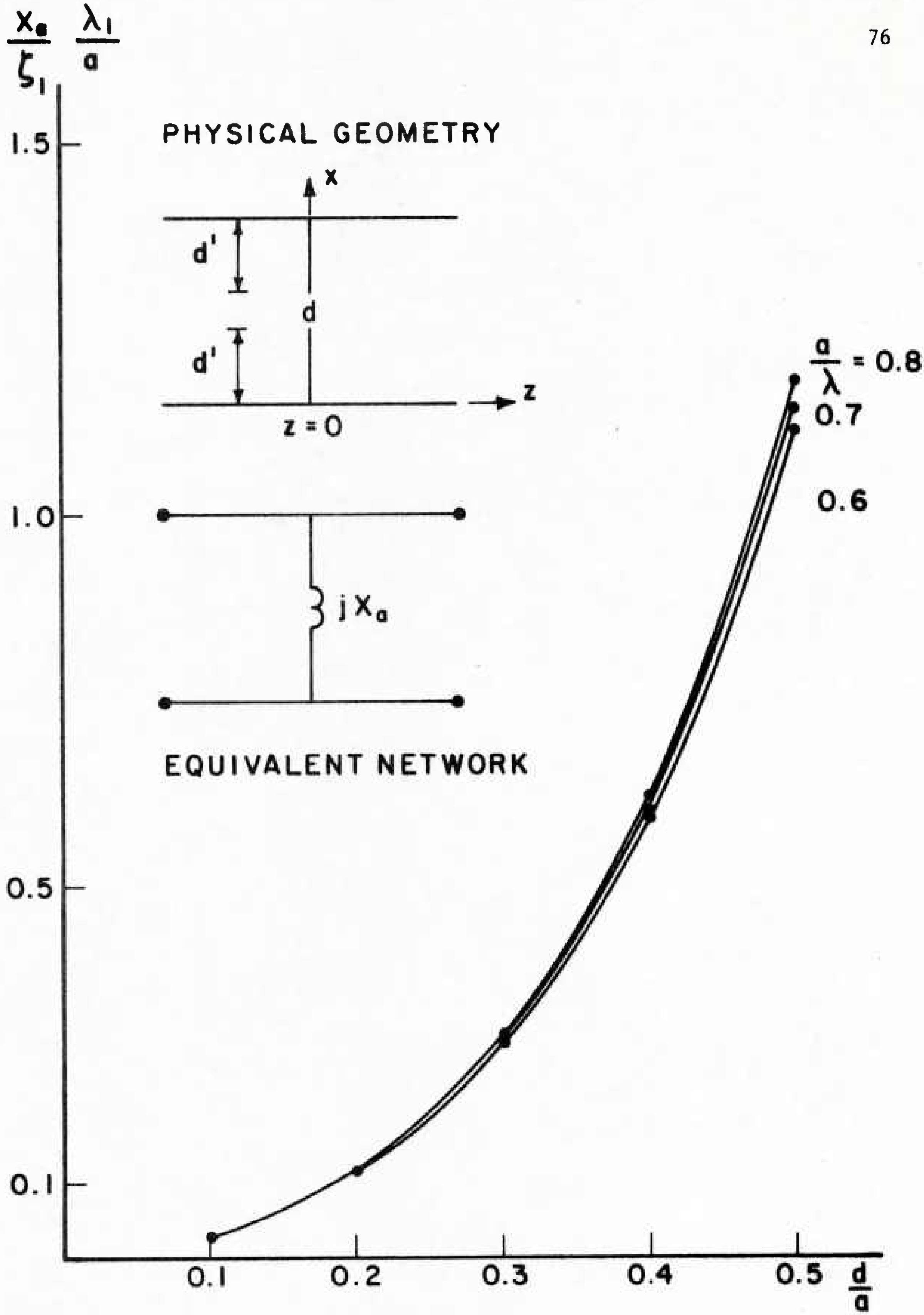

Figure 7. Network reactance of the symmetrical thin window. The number of segments per diaphragm is 24.

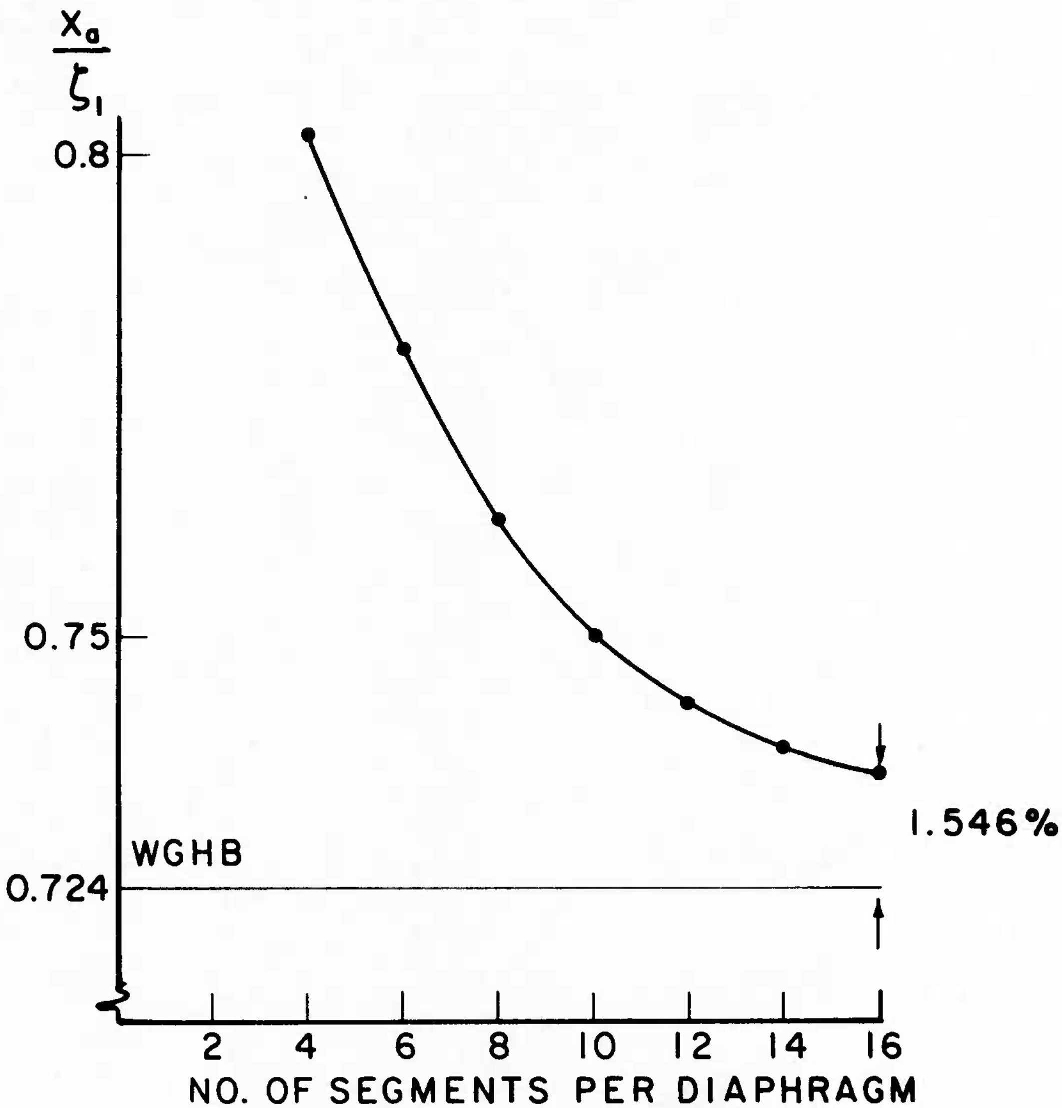

Figure 8. The convergence behavior for the symmetrical thin windows ($\frac{a}{\lambda}$ = 0.8, $\frac{d}{a}$ = 0.5).

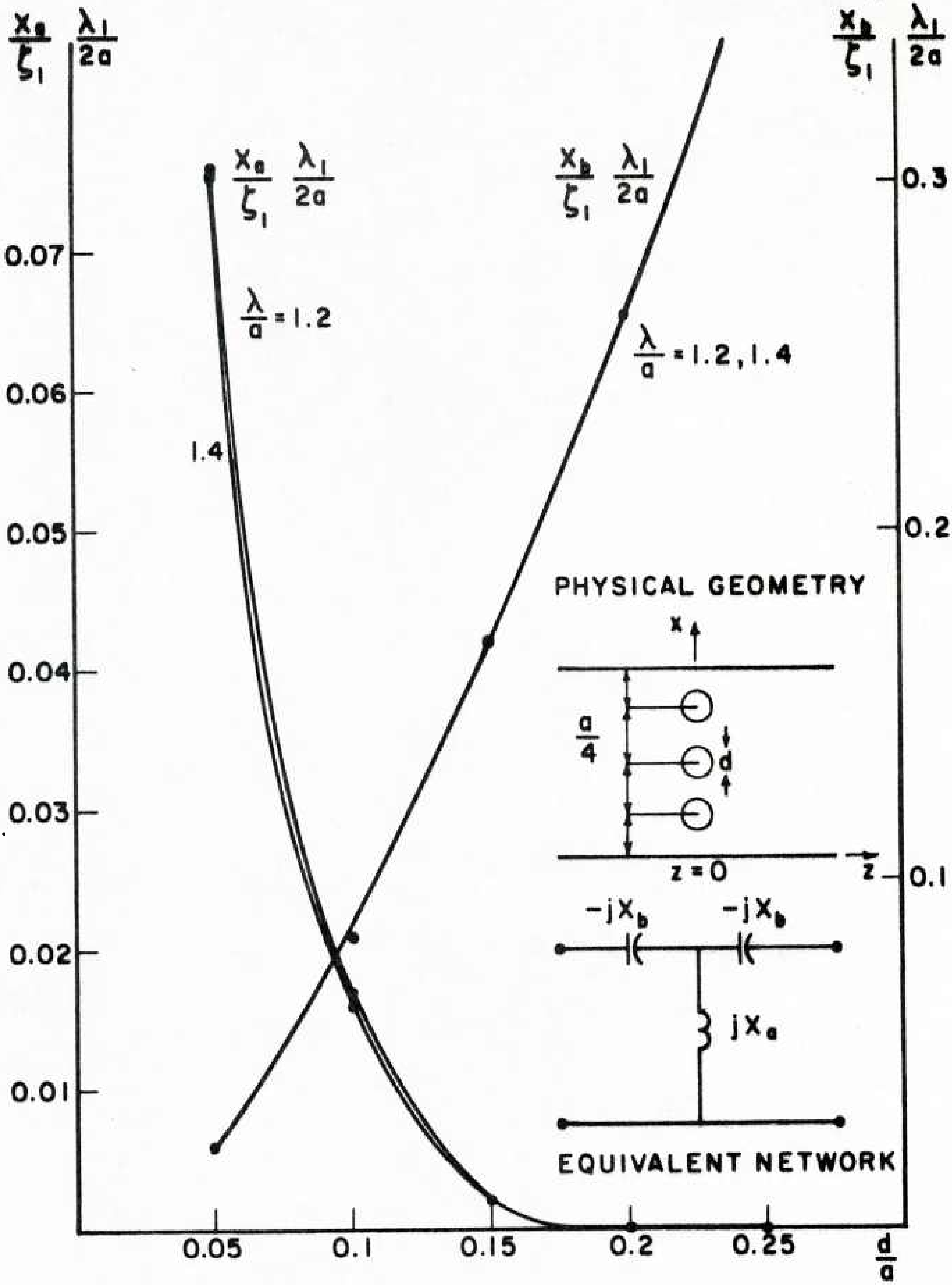

Figure 9. Network reactances of the symmetrical triple post. The number of segments used for each post is 16.

A complete field analysis of the problem is first given. From this analysis, the scattering and impedance matrix representations of the system of posts fully describing its effect on the dominant waveguide mode are obtained. Since the whole structure is both reciprocal and lossless, the scattering and impedance matrices are symmetric and unitary, and symmetric and pure imaginary, respectively. The latter is then realized in the form of a T-network of reactive elements.

The reactances of some post configurations are computed in Section 9, and more results can be found in [19]. In the actual computation, pulse expansions of the currents induced on the posts are used. Although chosen primarily so as to render the procedure economical, the choice is very natural, since pulses are instrumental in the definition of integration [16, Chapter 10]. This choice has proven very successful, nevertheless, as is evident by the performance of the solution.

The circular post problem was first treated using the Variational Method [1, Chapter 2], [2, Section 8-7]. The Variational Method was also applied to solve the problems of the inductive thin and thick irises [9, Sections 8-4 and 8-5], and of an array of symmetric thick irises [23]. Single and triple circular posts have been considered in [7], [8], where many of their characteristics have been discussed. The present analysis, however, is quite general. It can also become the first step in the solution of a system of dielectric posts in the inductive position in a rectangular waveguide.

Appendix A

Consider the function defined by the series

$$\left.\begin{aligned} G_1 &= \sum_{n=1}^{\infty} \frac{\sin(n\eta)\ \sin(n\eta')e^{-n\sigma}}{n} \\ \sigma &= |\zeta - \zeta'| \ . \end{aligned}\right\} \tag{A.1}$$

Since

$$\sin(n\eta)\ \sin(n\eta') = \frac{1}{2}\ (\cos\ n(\eta-\eta') - \cos\ n(\eta+\eta'))$$

$$= \frac{1}{2}\ \mathrm{Re}(e^{-jn(\eta-\eta')} - e^{-jn(\eta+\eta')}) \tag{A.2}$$

(A.1) becomes

$$G_1 = \frac{1}{2}\ \mathrm{Re}(\sum_{n=1}^{\infty} \frac{1}{n}\ e^{-n(\sigma+j(\eta-\eta'))} - \sum_{n=1}^{\infty} \frac{1}{n}\ e^{-n(\sigma+j(\eta+\eta'))}). \tag{A.3}$$

In (A.2) and (A.3), Re(z) denotes the real part of z.

Since

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad z \in C,\ |z| < 1 \tag{A.4}$$

and the series in (A.4) converges uniformly for all z, $|z| \leq |z_o| < 1$, a term by term integration can be carried out [20, Section 5-4], giving

$$\int_0^{z_o} \frac{dz}{1-z} = \sum_{n=0}^{\infty} \int_0^{z_o} z^n\ dz$$

$$-\log\ (1-z_o) = \sum_{n=0}^{\infty} \frac{z_o^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{z_o^n}{n} \quad . \tag{A.5}$$

In (A.5), log denotes the natural logarith. Putting z_o equal to $e^{-(\sigma+j(\eta-\eta'))}$ and $e^{-(\sigma+j(\eta+\eta'))}$ in (A.5), then using the results in (A.3), G_1 becomes

$$G_1 = \frac{1}{2} \mathrm{Re}[\log (\frac{1 - e^{-(\sigma+j(\eta+\eta'))}}{1 - e^{-(\sigma+j(\eta-\eta'))}})] . \tag{A.6}$$

Finally, since

$$\left| \frac{1 - e^{-(\sigma+j(\eta+\eta'))}}{1 - e^{-(\sigma+j(\eta-\eta'))}} \right|^2 = \frac{(e^{\sigma} - \cos(\eta+\eta'))^2 + \sin^2(\eta+\eta')}{(e^{\sigma} - \cos(\eta-\eta'))^2 + \sin^2(\eta-\eta')}$$

$$= \frac{e^{2\sigma} - 2\, e^{\sigma} \cos(\eta+\eta') + 1}{e^{2\sigma} - 2\, e^{\sigma} \cos(\eta-\eta') + 1}$$

$$= \frac{\cosh \sigma - \cos\, (\eta+\eta')}{\cosh \sigma - \cos\, (\eta-\eta')} \tag{A.7}$$

G_1 is given by

$$G_1 = \frac{1}{4} \log (\frac{\cosh \sigma - \cos(\eta+\eta')}{\cosh \sigma - \cos(\eta-\eta')}) . \tag{A.8}$$

As σ tends to zero, G_1 becomes

$$G_1 = \frac{1}{4} \log (\frac{1 - \cos(\eta+\eta')}{1 - \cos(\eta-\eta')}) . \tag{A.9}$$

For all (η',ξ') in a small neighborhood of (η,ξ), the following approximation is valid:

$$\mathrm{Re}[\log\,(1 - e^{-(|\xi-\xi'| + j(\eta-\eta'))})]$$

$$\simeq \mathrm{Re}[\log\,(|\xi-\xi'| + j(\eta-\eta'))]$$

$$= \log\,(\sqrt{(\eta-\eta')^2 + (\xi-\xi')^2})\;. \qquad \text{(A.10)}$$

Thus, G_1 exhibits a logarithmic singularity of the form (A.10) at $(\eta',\xi') = (\eta,\xi)$.

Appendix B

Consider the series

$$\left.\begin{aligned} G_2 &= \sum_{n=2}^{\infty} a_n \sin(n\eta)\sin(n\eta') \\ a_n &= \frac{e^{-\beta_n|\xi-\xi'|}}{\beta_n} - \frac{e^{-n|\xi-\xi'|}}{n} \\ \beta_n &= \sqrt{n^2-\sigma^2}\ , \quad 1 < \sigma < 2\ . \end{aligned}\right\} \tag{B.1}$$

Clearly

$$|G_2| \leq \sum_{n=2}^{\infty} |a_n|\ . \tag{B.2}$$

Put

$$\left.\begin{aligned} a(t) &= \frac{e^{-\beta(t)|\xi-\xi'|}}{\beta(t)} - \frac{e^{-t|\xi-\xi'|}}{t} \\ \beta(t) &= \sqrt{t^2-\sigma^2}\ , \quad t \geq 2 \end{aligned}\right\} \tag{B.3}$$

$$F(z,t) = e^{-\beta(t)z} - e^{-tz}\ . \tag{B.4}$$

Then

$$a(t) = \int_{|\xi-\xi'|}^{\infty} F(z,t)\, dz\ . \tag{B.5}$$

Since $t < \beta(t)$ for all $t \geq 2$, then $a(t) > 0$ for all $t \geq 2$, and certainly so is a_n, $n=2,3,\ldots$.

That $\{a_n\}$ is a monotonically decreasing sequence follows from its positiveness. Since, F and $\frac{\partial}{\partial t}$ F are continuous for $t \geq 2$ and for all $z \geq |\xi-\xi'|$, and, furthermore, the integrals $\int_{|\xi-\xi'|}^{\infty} F(z,t)\, dz$ and $\int_{|\xi-\xi'|}^{\infty} \frac{\partial}{\partial t} F(z,t)\, dz$ converge uniformly, then [21, Section 7-5]

$$\frac{d}{dt} a(t) = \int_{|\xi-\xi'|}^{\infty} \frac{\partial}{\partial t} F(z,t)\, dz$$

$$= - \int_{|\xi-\xi'|}^{\infty} zt\, dz \left(\int_{z}^{\infty} F(u,t)\, du\right) \tag{B.6}$$

with the help of (B.3) and (B.4). Thus, $\frac{d}{dt} a(t) < 0$ on $[2, \infty)$. Consequently, a(t) is a monotonically decreasing function for all $t \geq 2$, and certainly so is a_n, $n=2,3,\ldots$.

Thus, G_2 is dominated by $\sum_{n=2}^{\infty} a_n$, a series of positive monotonically decreasing terms. Since

$$\beta_{n+1} > n\ , \qquad n = 2,3,\ \ldots \tag{B.7}$$

then

$$a_n < \frac{e^{-\beta_n|\xi-\xi'|}}{\beta_n} - \frac{e^{-\beta_{n+1}|\xi-\xi'|}}{\beta_{n+1}}, \quad n=2,3,\ldots\ . \tag{B.8}$$

Consequently

$$\sum_{n=2}^{N} a_n < \frac{e^{-\beta_2|\xi-\xi'|}}{\beta_2} - \frac{e^{-\beta_{N+1}|\xi-\xi'|}}{\beta_{N+1}} \tag{B.9}$$

whence

$$|G_2| \leq \sum_{n=2}^{\infty} a_n < \frac{e^{-\beta_2|\xi-\xi'|}}{\beta_2} \tag{B.10}$$

since $\frac{e^{-\beta_{N+1}|\xi-\xi'|}}{\beta_{N+1}}$ goes to zero as N goes to infinity. The rate of convergence of G_2 is therefore exponential.

Put

$$b_n = \frac{1}{\beta_n} - \frac{1}{n} , \qquad n = 2,3,\dots . \tag{B.11}$$

Then

$$a_n \leq b_n, \qquad n = 2,3,\dots . \tag{B.12}$$

Consequently

$$|G_2| \leq \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} b_n < \frac{1}{\beta_2} \tag{B.13}$$

which can be proven using a similar procedure. G_2 does therefore converge uniformly for all (η,ξ) and all (η',ξ').

All the above results can also be deduced from Figure B.1.

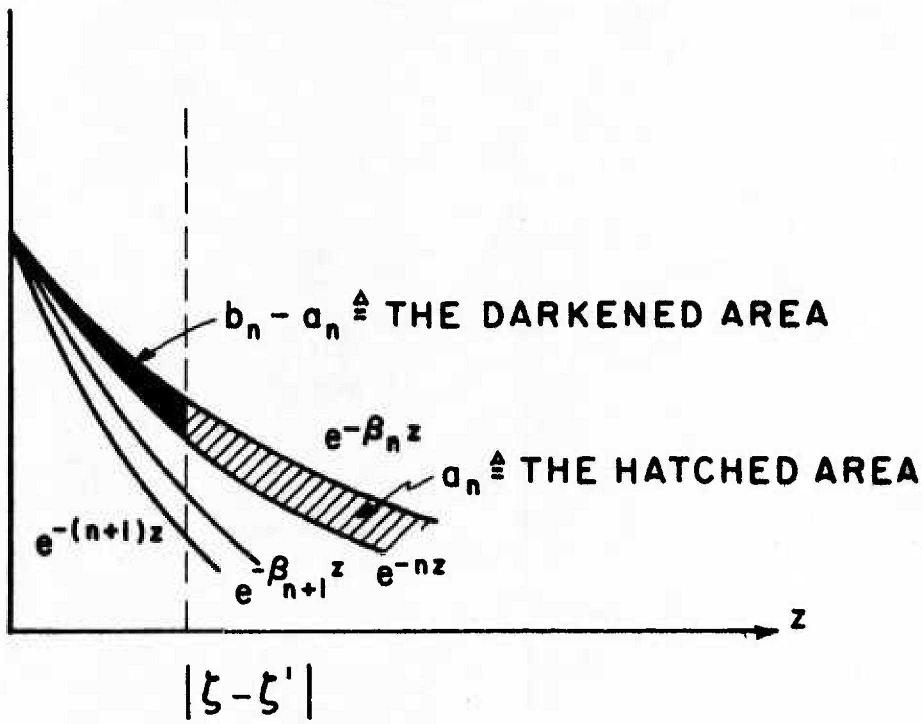

Figure B.1. Pictorial illustration of the procedure in the Appendix.

Chapter 4

MULTIPLE CAPACITIVE POSTS IN A RECTANGULAR WAVEGUIDE

Consider a system of posts $P^1, P^2, \ldots, P^p$ located close to each other in a rectangular waveguide. These posts are assumed perfectly conducting, of arbitrary shape and thickness, and uniform along the broad side of the waveguide, i.e., of the capacitive type. The medium filling the waveguide is assumed linear, homogeneous, isotropic, and dissipation free, and is therefore characterized by the real scalar permittivity ε and the real scalar permeability μ. The problem considered is depicted in Figure 1.

1. Preliminary Considerations

Let a TE_{10} to z mode of unit amplitude be incident on the posts from the left. This mode has the field distribution

$$\left.\begin{aligned} E_y^i &= \sin\left(\frac{\pi}{a}x\right) e^{-\gamma_0 z} \\ H_x^i &= \frac{-\gamma_0}{j\omega\mu} \sin\left(\frac{\pi}{a}x\right) e^{-\gamma_0 z} \\ H_z^i &= \frac{-\pi}{j\omega\mu a} \cos\left(\frac{\pi}{a}x\right) e^{-\gamma_0 z} \end{aligned}\right\} \tag{1}$$

where

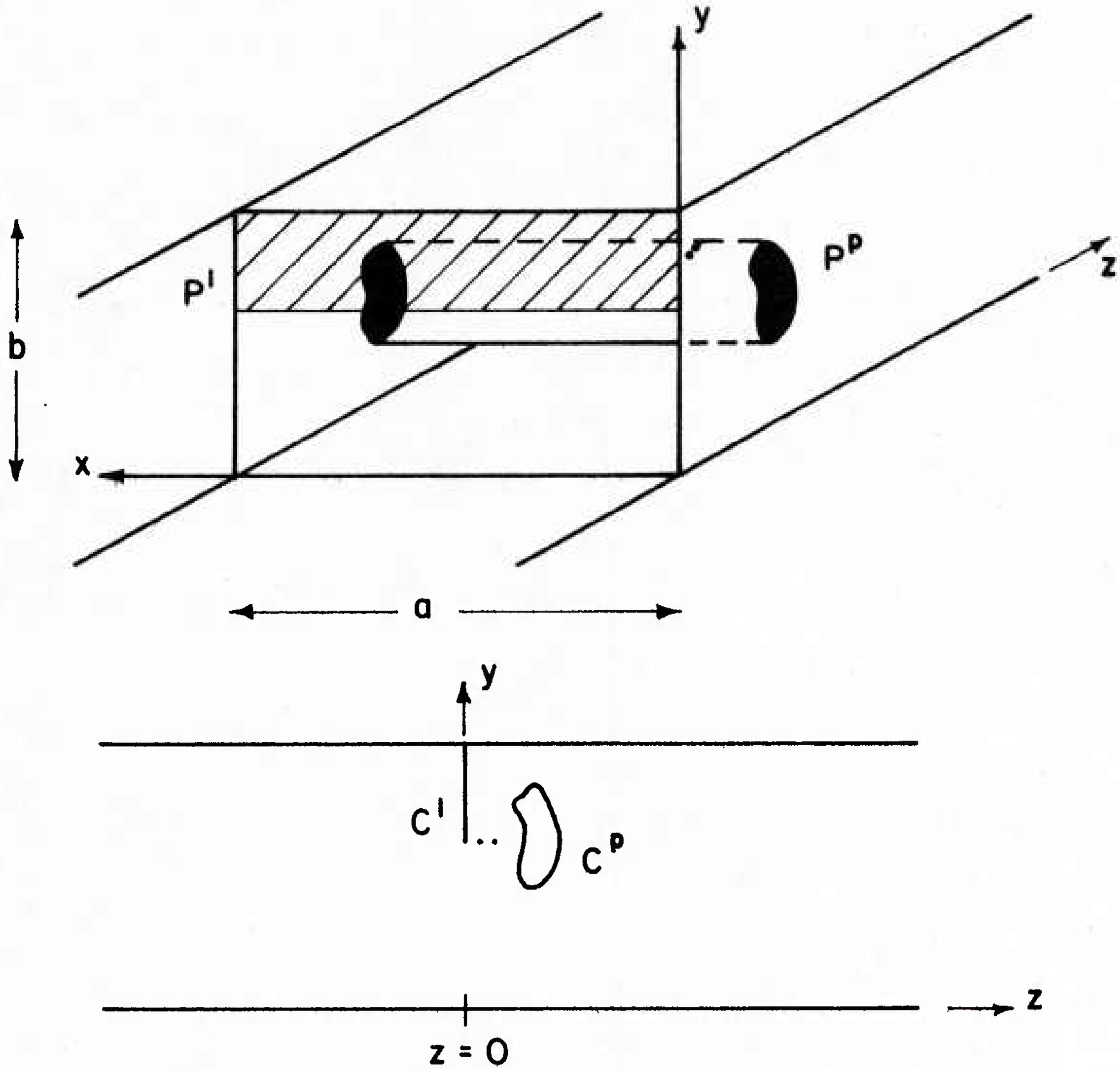

Figure 1. p capacitive posts in a rectangular waveguide.

$$\left.\begin{aligned} \gamma_0 &= j\,\frac{2\pi}{\lambda_0} = j\sqrt{\kappa^2 - (\frac{\pi}{a})^2} \\ \kappa &= \frac{2\pi}{\lambda} = \omega\sqrt{\mu\varepsilon} \ . \end{aligned}\right\} \tag{2}$$

In (2), κ is the wave number of the waveguide medium, and λ is its wave length. Furthermore, it is assumed that $a < \lambda < 2a$ and $2b < \lambda$ so that only the dominant mode can propagate in the waveguide.

Since each post is uniform along the x-axis, and since the exciting mode has no x-component of electric field, neither does the scattered field. That is, the scattered field is TE to x, and can therefore be derived from an electric vector potential $\underline{F}$ having only an x-component ψ [2, Section 8-7]:

$$F = \psi \underline{x} \ . \tag{3}$$

The scattered field is given in terms of ψ by

$$\left.\begin{aligned} \underline{E}^s &= -\nabla \times \psi\,\underline{x} \\ \underline{H}^s &= \frac{1}{j\omega\mu}\nabla \times \nabla \times \psi\,\underline{x} \end{aligned}\right\} \tag{4}$$

while ψ itself satisfies

$$(\nabla^2 + \kappa^2)\psi = 0 \ . \tag{5}$$

Expanding (4) in rectangular coordinates, the components of the scattered field are found to be

$$\left.\begin{aligned}
E_x^s &= 0 \\
E_y^s &= -\frac{\partial}{\partial z}\,\psi \\
E_z^s &= \frac{\partial}{\partial y}\,\psi \\
H_x^s &= \frac{1}{j\omega\mu}\left(\frac{\partial^2}{\partial x^2} + \kappa^2\right)\psi \\
H_y^s &= \frac{1}{j\omega\mu}\,\frac{\partial^2}{\partial x\partial y}\,\psi \\
H_z^s &= \frac{1}{j\omega\mu}\,\frac{\partial^2}{\partial x\partial z}\,\psi\ .
\end{aligned}\right\} \tag{6}$$

Furthermore, since each post is uniform along the x-axis, and since the exciting mode has an x-component of magnetic field that varies as sin $(\frac{\pi}{a}\,x)$, so does the scattered field. It then follows from (6) that ψ must contain sin $(\frac{\pi}{a}\,x)$ as its x-dependent factor. Thus

$$\psi = \sin\left(\frac{\pi}{a}\,x\right)\,\psi(y,z)\ . \tag{7}$$

Substituting (7) into (5) and using (2), it becomes

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \gamma_0^2\right)\psi(y,z) = 0\ . \tag{8}$$

The components of the scattered field are now given by

$$\left.\begin{aligned}
E_x^s &= 0 \\
E_y^s &= -\sin\left(\frac{\pi}{a}x\right)\frac{\partial}{\partial z}\psi(y,z) \\
E_z^s &= \sin\left(\frac{\pi}{a}x\right)\frac{\partial}{\partial y}\psi(y,z) \\
H_x^s &= \frac{-\gamma_0^2}{j\omega\mu}\sin\left(\frac{\pi}{a}x\right)\psi(y,z) \\
H_y^s &= \frac{\pi}{j\omega\mu a}\cos\left(\frac{\pi}{a}x\right)\frac{\partial}{\partial y}\psi(y,z) \\
H_z^s &= \frac{\pi}{j\omega\mu a}\cos\left(\frac{\pi}{a}x\right)\frac{\partial}{\partial z}\psi(y,z) \quad .
\end{aligned}\right\} \tag{9}$$

The total field, incident plus scattered, must have zero tangential electric field at the waveguide walls. The incident field is a free waveguide mode, and does therefore have zero electric field tangent to the walls. The scattered field must then have zero tangential electric field at the walls. This is readily accomplished by setting

$$\frac{\partial}{\partial y}\psi(y,z) = 0, \qquad y = 0,b, \text{ and all } x \text{ and } z \tag{10}$$

which are the required boundary conditions on the scattered field.

The boundary conditions (10), once satisfied for any value of x, are clearly satisfied for all values of x. Thus, the problem is basically a two-dimensional scalar one that can entirely be worked out in some x=constant plane within the waveguide. Considerable simplification in the solution can result from choosing the $x = \frac{a}{2}$ plane. In this plane, the only components of the scattered field are

$$\left.\begin{aligned} E_y^s &= -\frac{\partial}{\partial z}\,\psi(y,z) \\ E_z^s &= \frac{\partial}{\partial y}\,\psi(y,z) \\ H_x^s &= \frac{-\gamma_0^2}{j\omega\mu}\,\psi(y,z) \end{aligned}\right\} \qquad (11)$$

whereas those of the incident field are

$$\left.\begin{aligned} E_y^i &= e^{-\gamma_0 z} \\ H_x^i &= \frac{-\gamma_0}{j\omega\mu}\,e^{-\gamma_0 z} \quad . \end{aligned}\right\} \qquad (12)$$

Finally, an important fact can be established by realizing that the set of equations (8), (10), and (11) is the same as the set satisfied by the fields in a parallel plate transmission line, except for γ_0 replacing $j\kappa$. That is, in order to solve any problem of capacitive posts in a rectangular waveguide, one need solve only the parallel plate transmission line problem which has the same cross section, at the same time replacing $j\kappa$ by γ_0. In the latter problem, the exciting field is a TE_0 to x mode, again with γ_0 replacing $j\kappa$.

No further simplification is possible, but ψ is still to be determined. This can be accomplished with the help of the Green's function for TE_{1n} to x modes in a rectangular waveguide obtained in the next section.

2. The Green's Function for TE_{1n} to x Modes in a Rectangular Waveguide

Consider a magnetic current filament $\underline{M}$ directed across the waveguide parallel to the x-axis and located at (y', z'). Furthermore, $\underline{M}$ is assumed to vary as $\sin(\frac{\pi}{a} x)$ along the x-axis. Figure 2 shows the situation considered.

Since $\underline{M}$ is directed along the x-axis, the field produced must have an x-component of magnetic field and no x-component of electric field. Furthermore, since $\underline{M}$ varies as $\sin(\frac{\pi}{a} x)$ along the x-axis, so does H_x. This can readily be verified, for instance, from the reciprocity theorem [2, Section 3-8]. An electric vector potential having only an x-component ψ, proportional to H_x, as is seen in Section 1, can then be used to derive all field components.

Only TE_{1n} to x modes can be excited in the waveguide, since these are the only modes having an H_x component that varies as $\sin(\frac{\pi}{a} x)$ and no E_x component [2, Section 4-4]. The potential function ψ due to the filament, relabeled G, is therefore referred to as the Green's function for TE_{1n} to x modes in a rectangular waveguide. Below, G is found as a series of these modes.

Put

$$G = \sin(\frac{\pi}{a} x)\ G(y,z) \quad . \tag{13}$$

G(y,z) then satisfies, for each x, the wave equation

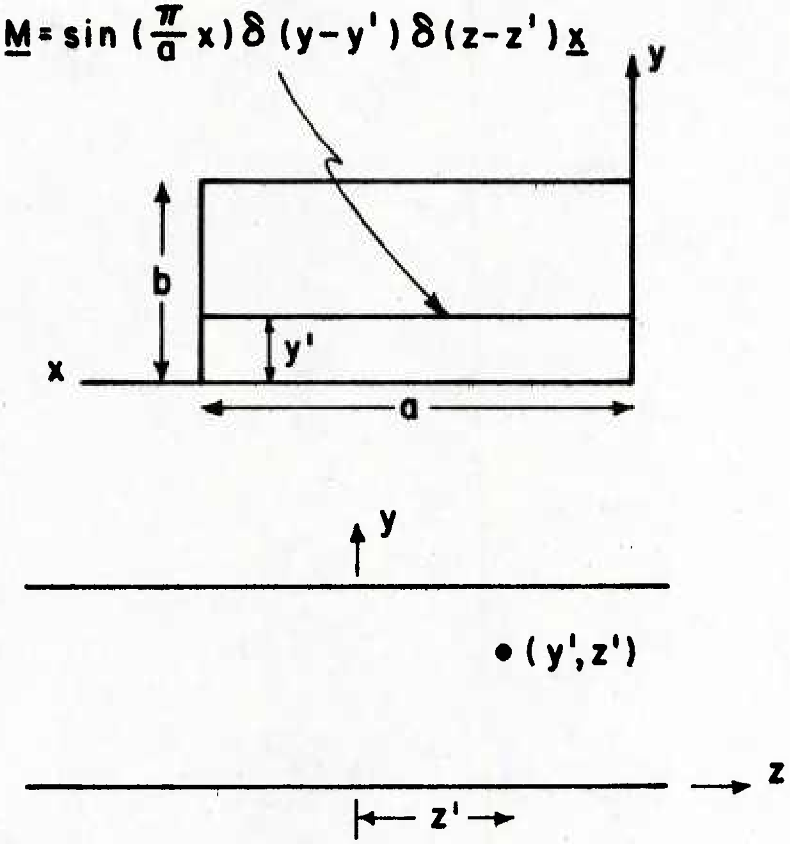

Figure 2. A magnetic current filament $\underline{M}$ in a rectangular waveguide parallel to the x-axis.

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \gamma_0^2\right) G(y,z) = - \delta(y-y') \delta(z-z') \tag{14}$$

together with the boundary conditions

$$\frac{\partial}{\partial y} G(y,z) = 0 \ , \qquad y = 0,b, \quad \text{and all } z. \tag{15}$$

In (14), δ is the Dirac delta function. Myltiplying both sides of (14) by $\cos\left(\frac{n\pi}{b} y\right)$, then integrating over y from 0 to b, it becomes

$$\left(\frac{d^2}{dz^2} - \gamma_0^2 - \left(\frac{n\pi}{b}\right)^2\right) \int_0^b G(y,z) \cos\left(\frac{n\pi}{b} y\right) dy$$

$$= - \cos\left(\frac{n\pi}{b} y'\right) \delta(z-z') \ . \tag{16}$$

Put

$$\left.\begin{aligned} G_n(z) &= \int_0^b G(y,z) \cos\left(\frac{n\pi}{b} y\right) dy \\ \gamma_n &= \sqrt{\left(\frac{n\pi}{b}\right)^2 + \gamma_0^2} \end{aligned}\right\} \quad n = 0,1,2,\ldots \ . \tag{17}$$

The one-dimensional wave equation (16) then becomes

$$\left(\frac{d^2}{dz^2} - \gamma_n^2\right) G_n(z) = - \cos\left(\frac{n\pi}{b} y'\right) \delta(z-z') \ . \tag{18}$$

For the solution of (18) to represent waves traveling away from the source, G_n must be of the form

$$G_n(z) = \begin{cases} A_n \, e^{-\gamma_n z} & z > z' \\ B_n \, e^{\gamma_n z} & z' > z \end{cases} \tag{19}$$

where A_n and B_n are constants to be determined. Since G is proportional to H_x, it is continuous across the filament at $z = z'$ [2, Section 1-14], and so is G_n. Thus

$$A_n e^{-\gamma_n z'} - B_n e^{\gamma_n z'} = 0 . \tag{20}$$

Furthermore, integrating (18) over z from $z' - \Delta$ to $z' + \Delta$, then letting Δ go to zero, there then results

$$\frac{d}{dz} G_n(z) \Big|_{z'_-}^{z'_+} = - \cos \left(\frac{n\pi}{b} y'\right) . \tag{21}$$

That is, $\frac{d}{dz} G_n$ is discontinuous at $z = z'$ by the amount $- \cos \left(\frac{n\pi}{b} y'\right)$. Thus

$$A_n \gamma_n e^{-\gamma_n z'} + B_n \gamma_n e^{\gamma_n z'} = \cos \left(\frac{n\pi}{b} y'\right) . \tag{22}$$

Solving (20) and (22) simultaneously, A_n and B_n are found to be

$$A_n = \frac{1}{2\gamma_n} \cos \left(\frac{n\pi}{b} y'\right) e^{\gamma_n z'} \tag{23}$$

$$B_n = \frac{1}{2\gamma_n} \cos \left(\frac{n\pi}{b} y'\right) e^{-\gamma_n z'} . \tag{24}$$

Combining (23) and (24) with (19), G_n becomes

$$G_n(z) = \frac{1}{2\gamma_n} \cos \left(\frac{n\pi}{b} y'\right) e^{-\gamma_n |z-z'|} , \quad n=0,1,2,\ldots . \tag{25}$$

By Fourier theory [15, Section 43], (17) can be inverted as

$$G(y,z|y',z') = \frac{1}{b}\sum_{n=0}^{\infty} \varepsilon_n \frac{\cos\left(\frac{n\pi}{b}y\right)\cos\left(\frac{n\pi}{b}y'\right)e^{-\gamma_n|z-z'|}}{2\gamma_n} \tag{26}$$

where

$$\varepsilon_n = \begin{cases} 1 & n = 0 \\ 2 & n \geq 1 \end{cases} . \tag{27}$$

Clearly, G satisfies the boundary conditions (15).

3. Basic Formulation

As is pointed out in Section 1, the problem is a two-dimensional scalar one that can entirely be worked out in the $x = \frac{a}{2}$ plane within the waveguide. Thus, all source and field points are, hereafter, assumed located in this plane. The incident field is then given by (12), whereas that scattered from the posts is given by (11).

Let $(\underline{E}^i, \underline{H}^i)$ be incident while all the posts are absent, and $(\underline{E}(\underline{J}), \underline{H}(\underline{J}))$ be the field produced by an electric current of density $\underline{J} = \bigcup_{m=1}^{P} \underline{J}^m$, where $\underline{J}^m$ is the current on C^m, while all the posts are absent. By the uniqueness theorem [2, Section 3-3], $(\underline{E}^i + \underline{E}(\underline{J}), \underline{H}^i + \underline{H}(\underline{J}))$ is identical with the original field whenever

$$\underline{n}^m \times (\underline{E}^i + \underline{E}(\underline{J})) = 0 \qquad \text{on } C^m . \tag{28}$$

In (28), $\underline{n}^m$ is the outward unit vector normal to C^m and tangent to the $x = \frac{a}{2}$ plane. $(\underline{E}(\underline{J}), \underline{H}(\underline{J}))$ must then have the field distribution (11). Since

$$\underline{n}^m \times \left(\underline{H}(\underline{J})\Big|_{\nu=0_+} - \underline{H}(\underline{J})\Big|_{\nu=0_-}\right) = \underline{J}^m \quad \text{on } C^m \tag{29}$$

where ν is the distance along $\underline{n}^m$ from C^m, and $\underline{H}(\underline{J})$ has only an x-component that does not vary with x, $\underline{J}^m$ has only an x-transverse component J^m that does not vary with x:

$$\underline{J}^m = J^m(y,\ z)\ \underline{t}^m = J^m(t)\ \underline{t}^m\ . \tag{30}$$

In (30), $\underline{t}^m$ is the counterclockwise unit vector along C^m, and t is the distance along C^m from an arbitrary, but fixed, point to (y,z).

Consider the magnetic current distribution

$$\underline{M}^m = -\frac{j\omega\mu}{\gamma_0^2}\,\nabla' \times J^m(t')\ \ \delta(\nu')\ \underline{t}^m \quad \text{on } C^m\ . \tag{31}$$

Since

$$\nabla' \times \phi\ \underline{t}^m = \nabla'\phi \times \underline{t}^m = \frac{\partial}{\partial\nu'}\ \phi\ \underline{n}^m \times \underline{t}^m = -\frac{\partial}{\partial\nu'}\ \ \underline{x} \tag{32}$$

for any function ϕ independent of x, (31) becomes

$$\underline{M}^m = \frac{j\omega\mu}{\gamma_0^2}\ \ J^m(t')\ \frac{d}{d\nu'}\ \delta(\nu')\ \underline{x} \quad \text{on } C^m\ . \tag{33}$$

The collection $\bigcup_{m=1}^{p} \underline{M}^m$ produces a field identical with $(\underline{E}(\underline{J}),\ \underline{H}(\underline{J}))$ [22]. By definition, $G(y,z|y',z')\ \underline{x}$ is the electric vector potential produced at any point (y,z) by a unit magnetic current filament in the x-direction located at (y', z'). Thus, by superposition, $\underline{J}$ produces at (y,z) the electric vector potential $\psi\,\underline{x}$, where

$$\left.\begin{aligned}\psi(y,z) &= \frac{j\omega\mu}{\gamma_0^2} \sum_{m=1}^{p} \int_{C^m} J^m(y',z') \left[\int_{-\nu_1'}^{\nu_2'} G(y,z|y',z') \frac{d}{d\nu'} \delta(\nu')\, d\nu' \right] dt' \\ dt' &= \sqrt{(dy')^2 + (dz')^2} \quad .\end{aligned}\right\} \tag{34}$$

In (34), G is given by (26) and (27), and primed and unprimed coordinates denote, respectively, source and field points. Whence

$$\psi = \frac{-j\omega\mu}{\gamma_0^2} \sum_{m=1}^{p} \int_{C^m} J^m\,(y',z') \frac{\partial}{\partial\nu'} G(y,z|y',z')\, dt' \ . \tag{35}$$

Since $G(y,z|y',z')$ is a solution of the homogeneous wave equation (8) for all $(y,z) \neq (y',z')$, so is ψ. Furthermore, ψ satisfies the boundary conditions (10) by virtue of (15). Thus, ψ, and consequently the complete field solution of the problem, can be found once all J^m are known. Using (4), (12), and (35), (28) becomes

$$\underline{n}^\ell \times [e^{-\gamma_0 z}\, \underline{y} + \frac{j\omega\mu}{\gamma_0^2} \nabla \times (\sum_{m=1}^{p} \int_{C^m} J^m(y',z') \frac{\partial}{\partial\nu'} G(y,z|y',z')\, dt')\, \underline{x}]$$

$$= 0 \ , \quad (y,z) \in C^\ell, \quad 1 \leq \ell \leq p \ . \tag{36}$$

Since

$$\underline{n}^\ell \times (\nabla \times \phi\, \underline{x}) = \underline{n}^\ell \times (\nabla\, \phi \times \underline{x}) = n^\ell \times (\frac{\partial}{\partial\nu}\, \phi\, \underline{n}^\ell \times \underline{x})$$

$$= - \frac{\partial}{\partial\nu}\, \phi\, \underline{x} \tag{37}$$

(36) can be put in the form

$$\sin(\theta(\underline{n}^{\ell},\underline{y}))\, e^{-\gamma_0 z} - \frac{j\omega\mu}{\gamma_0^2} \sum_{m=1}^{p} \int_{C^m} J^m(y',z') \frac{\partial^2}{\partial\nu\partial\nu'} G(y,z|y',z')\, dt'$$

$$= 0, \qquad (y,z) \in C^{\ell}, \qquad 1 \leq \ell \leq p \tag{38}$$

which is an integral equation for $\underline{J}$. In (38), $\theta(\underline{n}^{\ell},\underline{y})$ is the angle $\underline{n}^{\ell}$ makes with the y-axis at (y,z).

The higher order ($n \geq 1$) modes excited are evanescent, i.e., decay exponentially with distance from the posts. Thus, at distances sufficiently far from the posts, only the dominant mode (n=0) can exist. The reflection coefficient of the dominant mode is readily found from (11), (12), (26), (27), and (35) as

$$\Gamma = \frac{j\omega\mu}{2\gamma_0 b} \sum_{m=1}^{p} \int_{C^m} J^m(y',z') \sin\,(\theta(\underline{n}^m,\underline{y}))\, e^{-\gamma_0 z'}\, dt' \,. \tag{39}$$

The transmission coefficient of the dominant mode is then

$$T = 1 + \frac{j\omega\mu}{2\gamma_0 b} \sum_{m=1}^{p} \int_{C^m} J^m(y',z') \sin(\theta(\underline{n}^m,\underline{y}'))\, e^{\gamma_0 z'}\, dt' \tag{40}$$

4. The Scattering Matrix

Following Montgomery et al. [10, Section 5-14], the scattering matrix of the posts is defined as

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} . \tag{41}$$

In (41), S_{11} and S_{21} are, respectively, the amplitudes of the dominant mode reflected to the left and transmitted to the right of the posts due to an incident TE_{10} to z mode of unit amplitude from the left. Consequently, S_{11} and S_{21} are given by (39) and (40), respectively.

Similarly, S_{22} and S_{12} are, respectively, the reflection and transmission coefficients of a TE_{10} to z mode of unit amplitude incident on the posts from the right. In the $x = \frac{a}{2}$ plane, this mode has the field distribution

$$\left.\begin{aligned} E_y^i &= e^{\gamma_0 z} \\ H_x^i &= \frac{\gamma_0}{j\omega\mu} e^{\gamma_0 z} \quad . \end{aligned}\right\} \qquad (42)$$

The previous analysis carries through in this case. Thus, the scattered field is given by (11) and (35), but with $\underline{J}' = \bigcup_{m=1}^{p} \underline{J}'^m$, now replacing $\underline{J}$ in (35), determined by solving the integral equation

$$\sin\,(\theta(\underline{n}^\ell,\underline{y}))\; e^{\gamma_0 z} - \frac{j\omega\mu}{\gamma_0^2} \sum_{m=1}^{p} \int_{C^m} J'^m(y',z') \frac{\partial^2}{\partial\nu\partial\nu'} G(y,z|y',z')\; dt'$$

$$= 0\ , \quad (y,z) \in C^\ell, \quad 1 \le \ell \le p \qquad (43)$$

rather than (38). It then follows from (11), (26), (27) and (35) that

$$S_{22} = \frac{j\omega\mu}{2\gamma_0 b} \sum_{m=1}^{p} \int_{C^m} J'^m(y',z') \sin(\theta(\underline{n}^m, \underline{y}'))\, e^{\gamma_0 z'}\, dt' \tag{44}$$

$$S_{12} = 1 + \frac{j\omega\mu}{2\gamma_0 b} \sum_{m=1}^{p} \int_{C^m} J'^m(y',z') \sin(\theta(\underline{n}^m, \underline{y}'))\, e^{-\gamma_0 z'}\, dt' \tag{45}$$

The scattering matrix is both symmetric and unitary. That is

$$S = S^T$$

$$SS^H = S^H S = U \tag{46}$$

where T and H denote matrix transpose and Hermitian, respectively, and U is the identity matrix.

Let $(\underline{E}^1, \underline{H}^1)$ and $(\underline{E}^2, \underline{H}^2)$ be the z-transverse fields in the waveguide, sufficiently far from the posts, due to TE_{10} to z modes of arbitrary amplitudes c_1 and c_2 incident from the left and from the right of the posts, respectively. It then follows from (11), (12), (26), (27), (35), (39), (40), (42), (44), and (45) that

$$\underline{E}^1 = \begin{cases} c_1 \left(e^{-\gamma_0 z} + S_{11}\, e^{\gamma_0 z}\right) \underline{y} & z \ll 0 \\ c_1\, S_{21}\, e^{-\gamma_0 z}\, \underline{y} & z \gg 0 \end{cases}$$

$$\underline{H}^1 = \begin{cases} -\eta_0\, c_1 \left(e^{-\gamma_0 z} - S_{11}\, e^{\gamma_0 z}\right) \underline{x} & z \ll 0 \\ -\eta_0\, c_1\, S_{21}\, e^{-\gamma_0 z}\, \underline{x} & z \gg 0 \end{cases} \tag{47}$$

$$\underline{E}^2 = \begin{cases} c_2 \; S_{12} \; e^{\gamma_0 z} \; \underline{y} & z << 0 \\ c_2 \; (e^{\gamma_0 z} + S_{22} \; e^{-\gamma_0 z}) \; \underline{y} & z >> 0 \end{cases}$$

$$\underline{H}^2 = \begin{cases} \eta_0 \; c_2 \; S_{12} \; e^{\gamma_0 z} \; \underline{x} & z << 0 \\ \eta_0 \; c_2 \; (e^{\gamma_0 z} - S_{22} \; e^{-\gamma_0 z}) \; \underline{x} & z >> 0 \; . \end{cases}$$

In (47), η_0 is the characteristic admittance of the dominant waveguide mode:

$$\eta_0 = \frac{1}{\zeta_0} = \frac{\gamma_0}{j\omega\mu} \quad . \tag{48}$$

Let W be the curve enclosing the surface area of the waveguide in the $x = \frac{a}{2}$ plane between the $z = z_1$ and z_2 lines, for some $z_1 << 0$ and $z_2 >> 0$. The reciprocity theorem then states that

$$\int_W (\underline{E}^1 \times \underline{H}^2 - \underline{E}^2 \times \underline{H}^1) \cdot \underline{n} \; dt = 0 \tag{49}$$

where $\underline{n}$ is the outward unit vector normal to W and tangent to the $x = \frac{a}{2}$ plane. Substituting (47) into (49), there then results

$$2b \; c_1 \; S_{12} \; c_2 = 2b \; c_1 \; S_{21} \; c_2 \tag{50}$$

whence

$$S_{12} = S_{21} \quad . \tag{51}$$

The scattering matrix is symmetric if and only if the whole structure is reciprocal.

That S is unitary follows from conservation of power considerations. Let the two dominant modes be simultaneously incident on the posts from the left and from the right. The complex power scattered far to the left and to the right of the posts in the $x = \frac{a}{2}$ plane is basically

$$P_{sc} = b\eta_0 \left(|c_1 S_{11} + c_2 S_{12}|^2 + |c_1 S_{21} + c_2 S_{22}|^2\right) \tag{52}$$

whereas that incident is given by

$$P_{in} = b\eta_0 \left(|c_1|^2 + |c_2|^2\right) . \tag{53}$$

Since the structure is lossless, and since P_{in} and P_{sc} are real, they must be equal. Put

$$\vec{c} = \begin{bmatrix} c_1 \\ \\ c_2 \end{bmatrix} . \tag{54}$$

Then

$$b\eta_0 \, \vec{c}^{\,H} \vec{c} = b\eta_0 \, \vec{c}^{\,H} S^H S \, \vec{c} \tag{55}$$

or

$$S^H S = U .$$

5. The Impedance Matrix

Let TE_{10} to z modes of arbitrary amplitudes c_1 and c_2 be incident on the posts from the left and from the right, respectively.

Let v_1 and v_2 be, respectively, the amplitudes of the E_y

component far to the left and to the right of the posts referred to the z = 0 plane. It then follows from (47) that

$$v_1 = (1 + S_{11})\, c_1 + S_{12}\, c_2 \tag{57}$$

$$v_2 = S_{21}\, c_1 + (1 + S_{22})\, c_2 \,. \tag{58}$$

The choice of z = 0 as a reference plane is only a matter of convenience. In matrix form (57) and (58) become

$$\vec{v} = (U + S)\, \vec{c} \tag{59}$$

where

$$\vec{v} = \begin{bmatrix} v_1 \\ \\ v_2 \end{bmatrix} . \tag{60}$$

Similarly, let i_1 and i_2 be, respectively, the amplitudes of the H_x component far to the left and to the right of the posts extrapolated back to the z = 0 plane. Then

$$-\zeta_0\, i_1 = (1 - S_{11})\, c_1 - S_{12}\, c_2 \tag{61}$$

$$\zeta_0\, i_2 = -\, S_{21}\, c_1 + (1 - S_{22})\, c_2 \,. \tag{62}$$

In matrix form, (61) and (62) become

$$\zeta_0\, \vec{i} = (U - S)\, \vec{c} \tag{63}$$

where

$$\vec{i} = \begin{bmatrix} -i_1 \\ \\ i_2 \end{bmatrix} . \tag{64}$$

To relate to network theory, let $(v_1, -i_1)$ and (v_2, i_2) be the complex voltage-current pairs at the terminals of a two-port network [10, Section 4-5]. Then

$$\vec{v} = Z\, \vec{i} \tag{65}$$

where Z is the network impedance matrix. From (59) and (63), Z is readily found as

$$Z = \zeta_0\, (U + S)(U - S)^{-1} \tag{66}$$

Since S is symmetric, so is Z. Furthermore, since

$$\begin{aligned} Z = \zeta_0(U + S)(U - S)^{-1} &= \zeta_0(S^H S + S)(S^H S - S)^{-1} \\ &= \zeta_0(S^H + U)(S^H - U)^{-1} \\ &= -\, Z^H \end{aligned} \tag{67}$$

the elements of Z are pure imaginary. Thus

$$\left.\begin{aligned} Z &= j \begin{bmatrix} X_{11} & X_{12} \\ \\ X_{21} & X_{22} \end{bmatrix} \\ X_{12} &= X_{21} \quad . \end{aligned}\right\} \tag{68}$$

6. The Equivalent Network

The complete field is seldom needed. Rather, the effect of the posts on the dominant waveguide mode is what must accurately be discribed. From an engineering perspective, a description in terms of a network of lumped elements is preferred.

The effect of the posts on the dominant waveguide mode is fully described by the impedance matrix Z. Such a representation can be realized in the form of a two-port T-network [10, Section 4-5].

The characteristic impedances of the TE_{1n} to x modes are given by

$$\left.\begin{aligned} \zeta_n &= -j\zeta\kappa \frac{\sqrt{(\frac{n\pi}{b})^2 + (\frac{\pi}{a})^2 - \kappa^2}}{\kappa^2 - (\frac{\pi}{a})^2} \qquad n \geq 1 \\ \zeta &= \sqrt{\frac{\mu}{\varepsilon}} \; . \end{aligned}\right\} \qquad (69)$$

Since these modes are evanescent, the energy stored close to the posts, in view of (69), is predominantly electric. This effect can suitably be represented by a capacitor in the shunt arm of the network. The elements in the series arms are also capacitors to account for the change difference across the posts in the z direction. The equivalent network of the posts is shown in Figure 3.

7. Solution of the Intergal Equation

The integral equation (38) can be written in compact form as

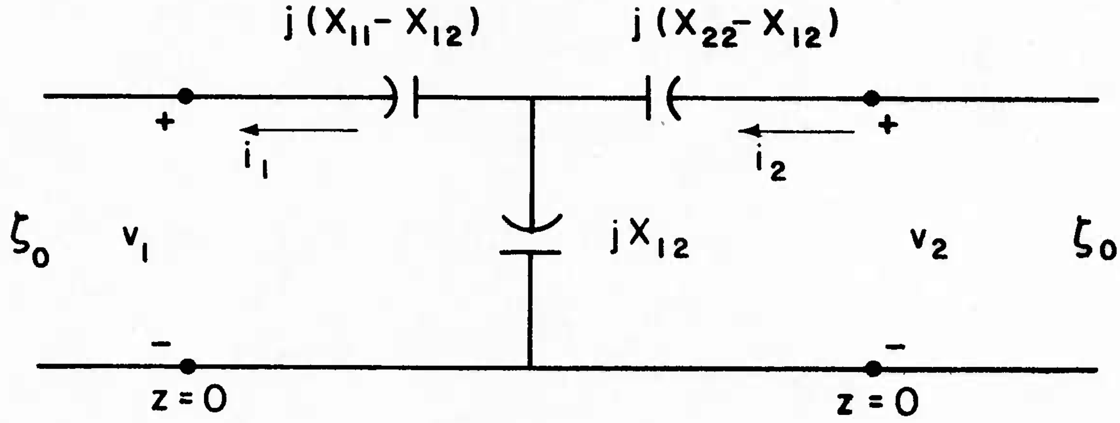

Figure 3. The equivalent network of the posts.

$$\left.\begin{aligned}
&\sum_{m=1}^{p} Z^m\,(J^m) = \sin\,(\theta(\underline{n}^{\ell},\ \underline{y}))\ V \\
&Z^m\,(J^m) = \frac{j\omega\mu}{\gamma_0^2}\int_{C^m} J^m(y',z')\ \frac{\partial^2}{\partial\nu\partial\nu'}\ G(y,z|y',z')\ dt' \\
&V = e^{-\gamma_0 z}\ , \qquad (y,\ z)\ \varepsilon\ C^{\ell}, \qquad 1 \leq \ell \leq p
\end{aligned}\right\} \tag{70}$$

An exact solution of (70) can rarely be obtained and an approximate solution has then to be sought.

Let each C^m be approximated by a polygon $\sum^m = \{S_1^m,\ S_2^m, \ldots, S_{q^m}^m\}$ as shown in Figure 4, and put

$$J^m\,(y',z') \simeq \sum_{j=1}^{q^m} I_j^m\ J_j^m(y',z')\ . \tag{71}$$

In (71), I_j^m are complex coefficients to be determined, whereas each J_j^m is a real function that vanishes on all $S_{i\,\neq\,j}^{\ell\,\neq\,m}$, but is otherwise unspecified. Thus, $\theta(\underline{n}^m,\ \underline{y})$ is constant on S_j^m:

$$\theta(\underline{n}^m,\ \underline{y}) = \theta(\underline{n}_j^m,\ \underline{y}) = \theta_j^m \qquad \text{on} \quad S_j^m\ . \tag{72}$$

Substituting (71) and (72) into (70), there then results

$$\sum_{m=1}^{p}\ \sum_{j=1}^{q^m} I_j^m\ Z_j^m(J_j^m) + r = \sin\,(\theta_i^{\ell})\ V\ ,$$

$$(y,z)\ \varepsilon\ S_i^{\ell}, \quad 1 \leq \ell \leq p,\ 1 \leq i \leq q^{\ell}. \tag{73}$$

The integrals in (73) are taken over S_j^m, and r is a residual term. A Galerkin solution [4, Section 1-3] can be obtained by requiring

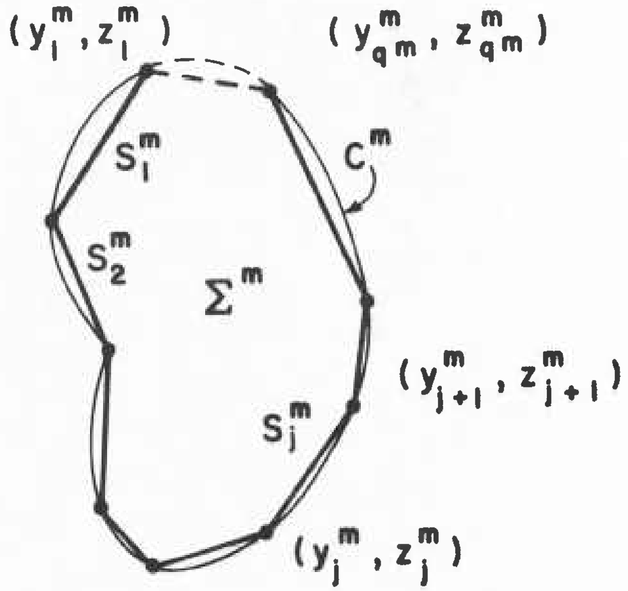

Figure 4. C^m approximated by a polygon $\sum^m$.

that r be orthogonal to all J_j^m.

Define the inner product

$$<A, B> = \int_{\bigcup_{\ell=1}^{p} \Sigma^\ell} A\, B^* \, dt \tag{74}$$

where * denotes complex conjugate. Taking the inner product of (73) with each J_i^ℓ and enforcing the Galerkin condition

$$<r, J_i^\ell> = 0 \qquad 1 \leq \ell \leq p \ , \quad 1 \leq i \leq q^\ell \tag{75}$$

it becomes

$$\sum_{m=1}^{p} \sum_{j=1}^{q^m} <I_j^m \ Z_j^m \ (J_j^m),\ J_i^\ell> = \sin\ (\theta_i^\ell)\ <V,\ J_i^\ell> \ ,$$

$$1 \leq \ell \leq p \quad , \quad 1 \leq i \leq q^\ell \ . \tag{76}$$

The system of equations (76) can be put in the matrix form

$$\bar{\bar{Z}}\ \vec{I} = D\ \vec{V} \ . \tag{77}$$

In (77), $\bar{\bar{Z}}$ is a p by p block matrix whose ℓmth block is the matrix

$$Z^{\ell m} = [Z_{ij}^{\ell m}]_{q^\ell \times q^m} = [<Z_j^m\ (J_j^m),\ J_i^\ell>]_{q^\ell \times q^m} \tag{78}$$

D is a p by p block diagonal matrix whose $\ell\ell$th block is the diagonal matrix

$$D^{\ell\ell} = [D_{ii}^{\ell\ell}]_{q^\ell \times q^\ell} = [\sin\ (\theta_i^\ell)]_{q^\ell \times q^\ell} \tag{79}$$

and $\vec{I}$ and $\vec{V}$ are p segment vectors whose mth and ℓth segments are the vectors

$$\vec{I}^m = [I_j^m]_{q^m \times 1} \tag{80}$$

$$\vec{V} = [V_i^\ell]_{q^\ell \times 1} = [\langle V, J_i^\ell \rangle]_{q^\ell \times 1} \tag{81}$$

respectively.

The currents J^m given by (71), with the coefficients I_j^m determined from (77), form the Galerkin solution of (70). A galerkin solution of (43) can be obtained in a similar manner. Clearly, then, using the same J_j^m, the solution is given by (71), but with coefficients now determined by solving (77) with the right-hand side vector $\vec{V}$ conjugated

8. Evaluation of the System of Equations

The construction of $\bar{\bar{Z}}$ in (77) constitutes a large portion of the work involved in the numerical solution. An efficient evaluation of the elements of $\bar{\bar{Z}}$ is therefore necessary for its success.

The ijth element of the ℓmth block is given by

$$Z_{ij}^{\ell m} = \frac{j\omega\mu}{\gamma_0^2} \int_{S_i^\ell} J_i^\ell\ (y,z)\ dt \int_{S_j^m} J_j^m\ (y',z')\ \frac{\partial^2}{\partial v \partial v'}\ G(y,z|y',z')\ dt' \tag{82}$$

where J_j^m are so far unspecified. A particularly simple choice for J_j^m is

$$J_j^m(y',z') = \begin{cases} 1 & (y',z') \in S_j^m \\ \\ 0 & (y',z') \in S_{i \neq j}^{\ell = m} \end{cases} \tag{83}$$

which corresponds to a pulse expansion of J_j^m. $Z_{ij}^{\ell m}$ then becomes

$$Z_{ij}^{\ell m} = \frac{j\omega\mu}{\gamma_0^2} \int_{S_i^\ell} dt \int_{S_j^m} \frac{\partial^2}{\partial\nu\partial\nu'} G(y,z|y',z')\, dt' \;. \tag{84}$$

Put

$$R_{ij}^{\ell m}(y,z) = \int_{S_j^m} \frac{\partial^2}{\partial\nu\partial\nu'} G(y,z|y',z')\, dt', \qquad (y,z) \in S_i^\ell \tag{85}$$

By the first mean value theorem of integration [16, Section 7-18], there exists a point $(y_o,z_o) \in S_i^\ell$ such that

$$Z_{ij}^{\ell m} = \frac{j\omega\mu}{\gamma_0^2} L_i^\ell R_{ij}^{\ell m} (y_o,z_o) \tag{86}$$

where

$$L_i^\ell = \sqrt{(y_{i+1}^\ell - y_i^\ell)^2 + (z_{i+1}^\ell - z_i^\ell)^2} \tag{87}$$

is the length of S_i^ℓ.

Put

$$G(y,z|y',z') = (G_0 + G_1 + G_2)(y,z|y',z') \;. \tag{88}$$

In (88)

$$\left.\begin{aligned}
G_0(y,z|y',z') &= \frac{e^{-j\frac{\pi}{b}|z-z'|\beta_0}}{j2\pi\beta_0} \\
G_1(y,z|y',z') &= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(\frac{n\pi}{b}y\right)\cos\left(\frac{n\pi}{b}y'\right)e^{-\frac{n\pi}{b}|z-z'|} \\
G_2(y,z|y',z') &= \frac{1}{\pi}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi}{b}y\right)\cos\left(\frac{n\pi}{b}y'\right) \\
&\quad\times\left(\frac{e^{-\frac{\pi}{b}|z-z'|\beta_n}}{\beta_n}-\frac{e^{-\frac{n\pi}{b}|z-z'|}}{n}\right)
\end{aligned}\right\} \tag{89}$$

$$\left.\begin{aligned}
\gamma_0 &= j\sqrt{\kappa^2-\left(\frac{\pi}{a}\right)^2} = j\frac{\pi}{b}\sqrt{\left(\frac{2b}{\lambda}\right)^2-\left(\frac{b}{a}\right)^2} = j\frac{\pi}{b}\beta_0 \\
\gamma_n &= \sqrt{\left(\frac{n\pi}{b}\right)^2+\gamma_0^2} = \frac{\pi}{b}\sqrt{n^2-\gamma_0^2} = \frac{\pi}{b}\beta_n\ .
\end{aligned}\right\} \tag{90}$$

The decomposition (88) therefore amounts to expressing the dynamic Green's function G in terms of a dominant mode wave G_0, the corresponding static Green's function G_1, which can be obtained from (26) by dropping the $n = 0$ term and setting γ_0 equal to zero in the remaining terms, plus a correction series G_2.

The series defining G_1 is readily summed to give

$$\begin{aligned}
G_1(y,z|y',z') &= \frac{1}{2b}|z-z'| \\
&\quad - \frac{1}{4\pi}\log\left[\cosh\left(\frac{\pi}{b}|z-z'|\right)-\cos\left(\frac{\pi}{b}(y+y')\right)\right) \\
&\quad \times\left(\cosh\left(\frac{\pi}{b}|z-z'|\right)-\cos\left(\frac{\pi}{b}(y-y')\right)\right)\Big]
\end{aligned} \tag{91}$$

where log denotes the natural logarithm. The details of the summation are given in Appendix A. The series in G_2 is dominated by an exponentially convergent series of positive monotonically decreasing terms, as is shown in Appendix B of Chapter 3, and can therefore be summed directly at a minimal cost.

The normal derivatives of G at $(y_o,z_o) \in S_i$, and at any $(y',z') \in S_j^m$ can be computed numerically through finite differences. Thus, for all $(\ell,i) \neq (m,j)$,

$$\frac{\partial^2}{\partial\nu\partial\nu'} G(y_o,z_o|y',z') \simeq \frac{1}{cc'} \sum_{k=1}^{M} \sum_{k'=1}^{M'} b_k b'_{k'} (G_0+G_1+G_2)(y_k,z_k|y'_{k'},z'_{k'}) . \tag{92}$$

Here, (y_k,z_k) are the pivot points along the normal to S_i^ℓ at (y_o,z_o), and b_k are the coefficients and c is the multiplier factor of the Mth difference formula. The parameters for the M'th difference formula, $(y'_{k'},z'_{k'})$, $b'_{k'}$, and c', are similarly defined.

When dealing with the diagonal elements of $\bar{\bar{Z}}$, $((\ell,i) = (m,j))$, G_1 offers a logarithmic singularity at (y_o,z_o) that requires particular attention. In Appendix A, the singular part of G_1 is found to be

$$G_{1s}(y_o,z_o|y',z') = \frac{-1}{4\pi} \log[2(\cosh(\frac{\pi}{b}(z_o-z')) - \cos(\frac{\pi}{b}(y_o-y')))] . \tag{93}$$

Put

$$G_{1p}(y_o,z_o|y',z') = (G_1 - G_{1s})(y_o,z_o|y',z') . \tag{94}$$

Then

$$\frac{\partial^2}{\partial\nu\partial\nu'} G(y_o,z_o|y',z') = \frac{1}{8b}\left(\frac{\pi}{b}\right) \mathrm{Re}\left[\frac{e^{j2\theta_i^\ell}}{\sinh^2\left(\frac{\pi}{b}\frac{\sigma}{2}\right)}\right]$$

$$+ \frac{\partial^2}{\partial\nu\partial\nu'} (G_0+G_{1p}+G_2)(y_o,z_o|y',z')$$

$$\simeq \frac{1}{8b}\left(\frac{\pi}{b}\right) \mathrm{Re}\left[\frac{e^{j2\theta_i^\ell}}{\sinh^2\left(\frac{\pi}{b}\frac{\sigma}{2}\right)}\right]$$

$$+ \frac{1}{cc'} \sum_{k=1}^{M} \sum_{k'=1}^{M'} b_k b'_{k'} (G_0+G_{1p}+G_2)(y_k,z_k|y'_{k'},z'_{k'}) \tag{95}$$

where

$$\sigma = (z_o-z') + j(y_o-y') \ . \tag{96}$$

The truth of the first term on the right-hand side of (95) can readily be verified in a straightforward manner by carrying out the normal differentiations, and Re(z) denotes the real part of z.

The evaluation of $Z_{ij}^{\ell m}$ is now completed by integrating $\frac{\partial^2}{\partial\nu\partial\nu'} G$ over S_j^m, and for that purpose any quadrature rule can be used. Thus,

$$Z_{ij}^{\ell m} = \frac{j\omega\mu}{\gamma_0^2} \frac{L_i^\ell L_j^m}{2} \sum_{u=1}^{N} q_u$$

$$\times \left[\frac{1}{cc'} \sum_{k=1}^{M} \sum_{k'=1}^{M'} b_k b'_k (G_0+G_1+G_2)(y_k,z_k|y'_{uk'},z'_{uk'})\right] \ . \tag{97}$$

In (97), $(y'_{uk'}, z'_{uk'})$ are the pivot points of the M' difference formula along the normal to S_j^m at $((1-p_u)y_j^m + p_u y_{j+1}^m, (1-p_u)z_j^m + p_u z_{j+1}^m)$, and q_u are the coefficients and p_u determine the location of the abscissas of the Nth integration rule. The diagonal elements of $\bar{\bar{Z}}$ are then given by

$$Z_{ii}^{\ell\ell} = \frac{j\omega\mu}{\gamma_0^2} L_i^\ell [G_{ii}^{\ell\ell}$$

$$+ \frac{L_i^\ell}{2} \sum_{u=1}^{N} q_u [\frac{1}{cc'} \sum_{k=1}^{M} \sum_{k'=1}^{M'} b_k b'_{k'} (G_0 + G_{1p} + G_2)(y_k, z_k | y'_{uk'}, z'_{uk'})]] \tag{98}$$

where

$$G_{ii}^{\ell\ell} = \frac{1}{8b} (\frac{\pi}{b}) \int_{S_i^\ell} Re[\frac{e^{j2\theta_i^\ell}}{\sinh^2 (\frac{\pi}{b} \frac{\sigma}{2})}] \, dt' . \tag{99}$$

The singularity in the integrand at (y_o, z_o) is not integrable. However, the limit of the integral as (y,z) approaches (y_o, z_o) along the normal at (y_o, z_o), not its value at (y_o, z_o), is what is actually needed. The integral is elementary, nevertheless, and is readily evaluated to give

$$G_{ii}^{\ell\ell} = \frac{-1}{4b} Re[e^{j\theta_i^\ell} (\coth(\frac{\pi}{b} \frac{L_i - L}{2} e^{j\theta_i^\ell}) + \coth(\frac{\pi}{b} \frac{L}{2} e^{j\theta_i^\ell}))] \tag{100}$$

where L is the distance from (y_i^ℓ, z_i^ℓ) to (y_o, z_o).

The ith element of the ℓth segment of $\vec{V}$ is given by

$$V_i^\ell = \int_{S_i^\ell} J_i^\ell(y,z)\, e^{-\gamma_0 z}\, dt \tag{101}$$

which, upon using (83), becomes

$$V_i^\ell = \int_{S_i^\ell} e^{-\gamma_0 z}\, dt\ . \tag{102}$$

The integration in (102) can be carried out exactly. However, there exists a point $(\tilde{y}_o,\tilde{z}_o)$ such that

$$V_i^\ell = L_i^\ell\, e^{-\gamma_0 \tilde{z}_o}\ . \tag{103}$$

Actually, finding such points (y_o,z_o) and $(\tilde{y}_o,\tilde{z}_o)$ is at least as difficult as computing the integrals themselves. For sufficiently small L_i^ℓ, however, the mid-point of S_i^ℓ can replace these points while introducing only very little error. The system of equations thus obtained, clearly, is one that results from enforcing the point matching condition

$$r(y,z) = 0\ ,$$

$$(y,z) \in \{(\frac{y_{i+1}^\ell + y_i^\ell}{2},\ \frac{z_{i+1}^\ell + z_i^\ell}{2}\ |1 \leq \ell \leq p,\ 1 \leq i \leq q^\ell\} \tag{104}$$

in (73), rather than the Galerkin condition (75).

9. Numerical Results

The solution procedure presented is readily translated into a computer program. The elements of the scattering matrix and the reactances of the equivalent T- or Π-network are basically the parameters to be computed.

The scattering parameters, thanks to (71), (72), and (83), are computed by

$$\left.\begin{aligned}
S_{11} &= \frac{j\omega\mu}{2\gamma_0 b} \sum_{m=1}^{p} \sum_{j=1}^{q^m} I_j^m \sin(\theta_j^m) \int_{S_j^m} e^{-\gamma_0 z'}\, dt' \\
S_{21} &= 1 + \frac{j\omega\mu}{2\gamma_0 b} \sum_{m=1}^{p} \sum_{j=1}^{q^m} I_j^m \sin(\theta_j^m) \int_{S_j^m} e^{\gamma_0 z'}\, dt' \\
S_{22} &= \frac{j\omega\mu}{2\gamma_0 b} \sum_{m=1}^{p} \sum_{j=1}^{q^m} I_j'^m \sin(\theta_j^m) \int_{S_j^m} e^{\gamma_0 z'}\, dt' \\
S_{12} &= 1 + \frac{j\omega\mu}{2\gamma_0 b} \sum_{m=1}^{p} \sum_{j=1}^{q^m} I_j'^m \sin(\theta_j^m) \int_{S_j^m} e^{-\gamma_0 z'}\, dt'
\end{aligned}\right\} \quad (105)$$

where I_j^m and $I_j'^m$ are the solutions of (77) with right-hand side vectors $\vec{V}$ and $\vec{V}^*$, respectively. The impedance matrix is then computed through (66). In carrying out the integrations in (105), and also in (97) and (98), an eight-point Gauss-Radau quadrature rule [17] is used. The parameters of the integration rule are shown in Table 1 of Chapter 3. For differentiation, a symmetric finite difference formula with $M = 2$, $b_1 = 1$, and $b_2 = -1$ is used. The multiplier factor c is set equal to two tenths of the length of the segment where differentiation occurs.

Because of the approximations involved in the solution, the scattering matrix need no longer be symmetric nor unitary. To determine the impedance matrix, S_{12} and S_{21} are first replaced by their average

$$S_{av} = \frac{1}{2}(S_{12} + S_{21}). \tag{106}$$

The impedance matrix can then have a non-zero real part.

To test the solution, the computer program is run for a few selected problems. In particular, the problems of the circular post and of the symmetrical thin strip are considered. Some of the results obtained are plotted in Figures 5-8.

In all the cases, the convergence for capacitive susceptance B_b is monotonic and from above, as can be seen for the thin strip in Figure 8. The computed susceptances are found to agree well with the data in the Waveguide Handbook [3], with only a few segments needed even for large posts. A complete assessment of the solution should also consider the (Frobenius) norm of the real part of $Y = Z^{-1}$ and the modulus of difference in transmission coefficients. These two numbers are computed in all program runs, and are usually $O(10^{-7})$.

10. Concluding Remarks

The system of capacitive posts in a rectangular waveguide, i.e., of metallic obstacles that are uniform along the broad side of the waveguide, but are otherwise of arbitrary shape and thickness, has been considered in this chapter.

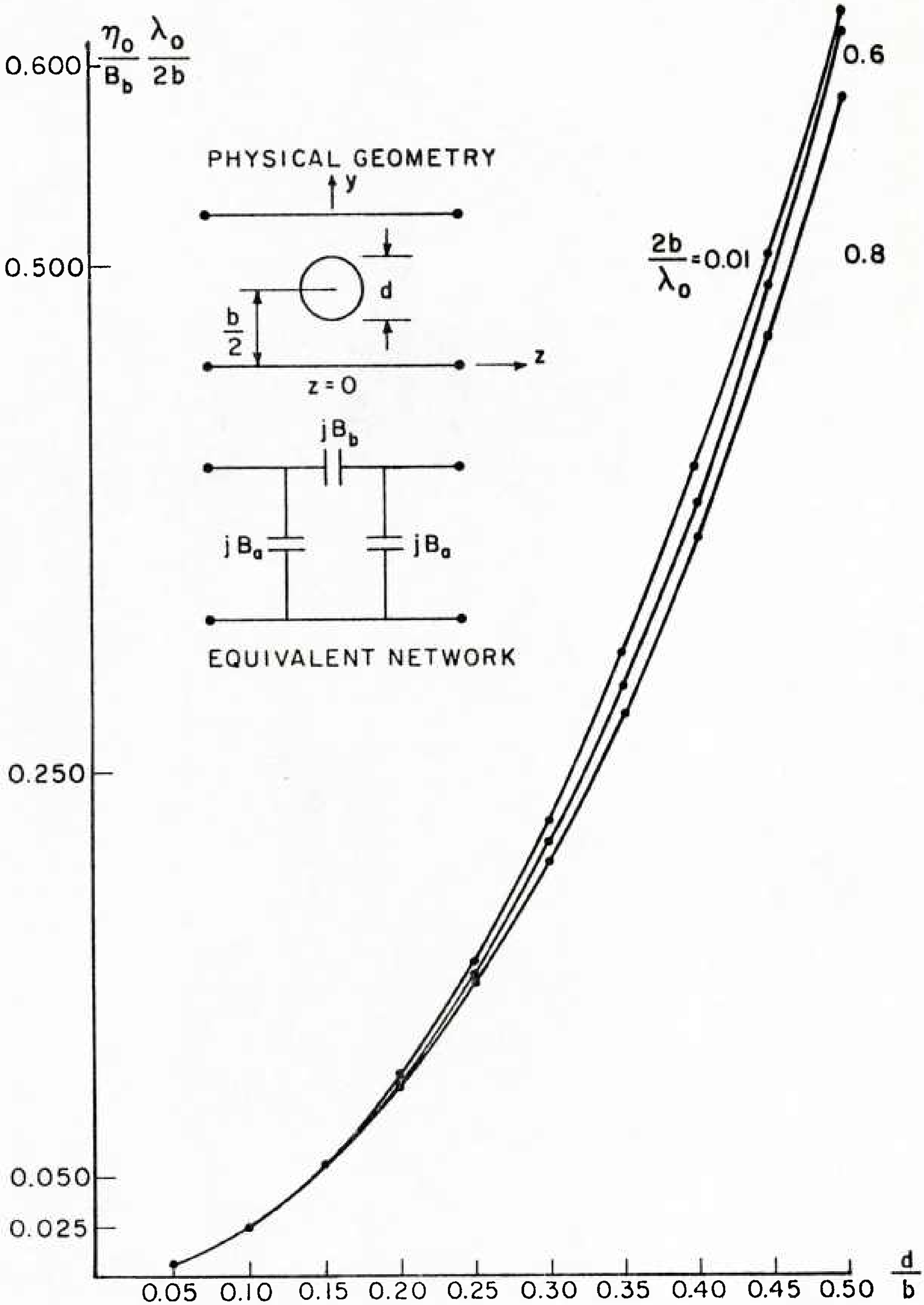

Figure 5. The series susceptance of the centered circular post. The number of segments used is 30.

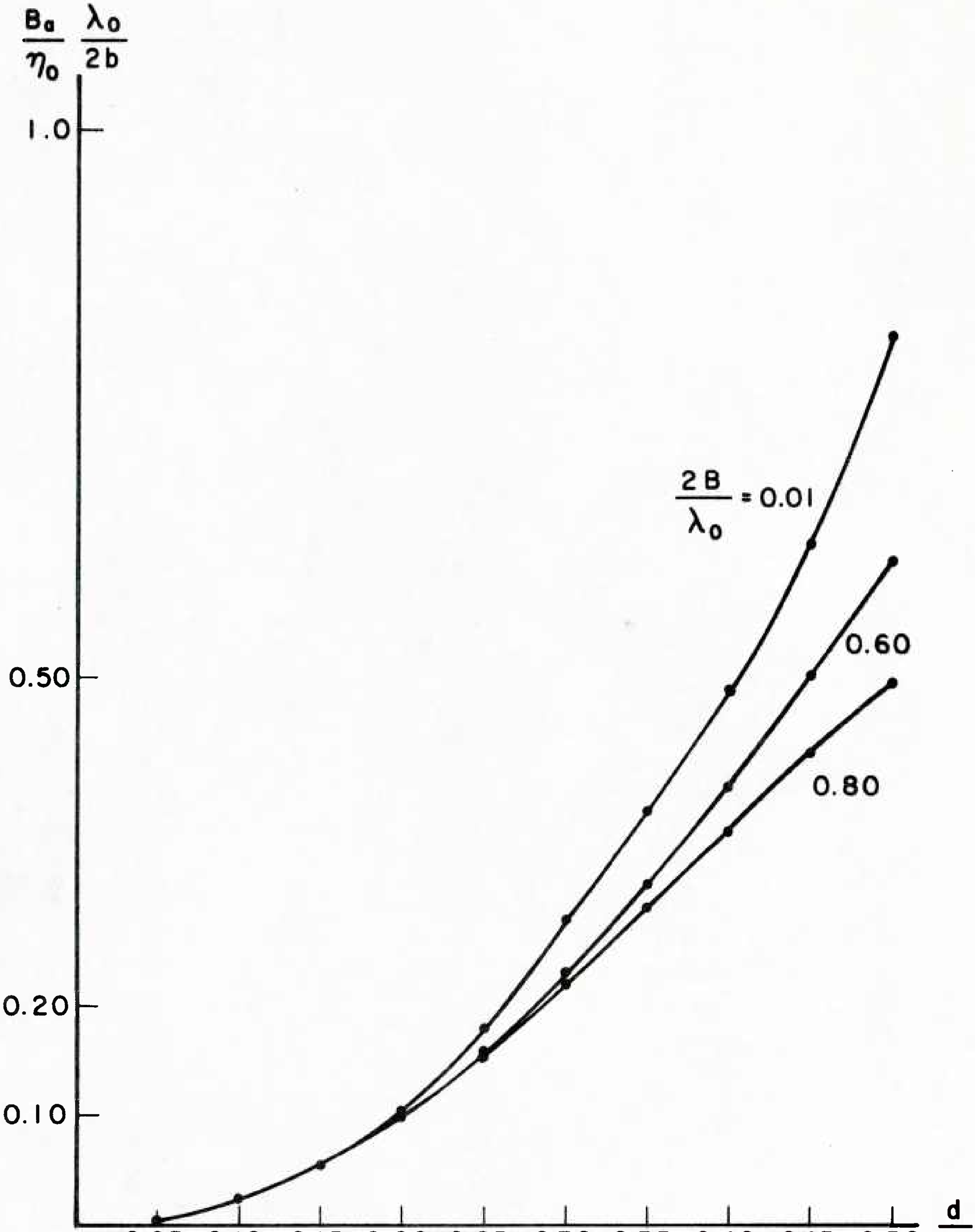

Figure 6. The parallel susceptance of the centered circular post. The number of segments used is 30.

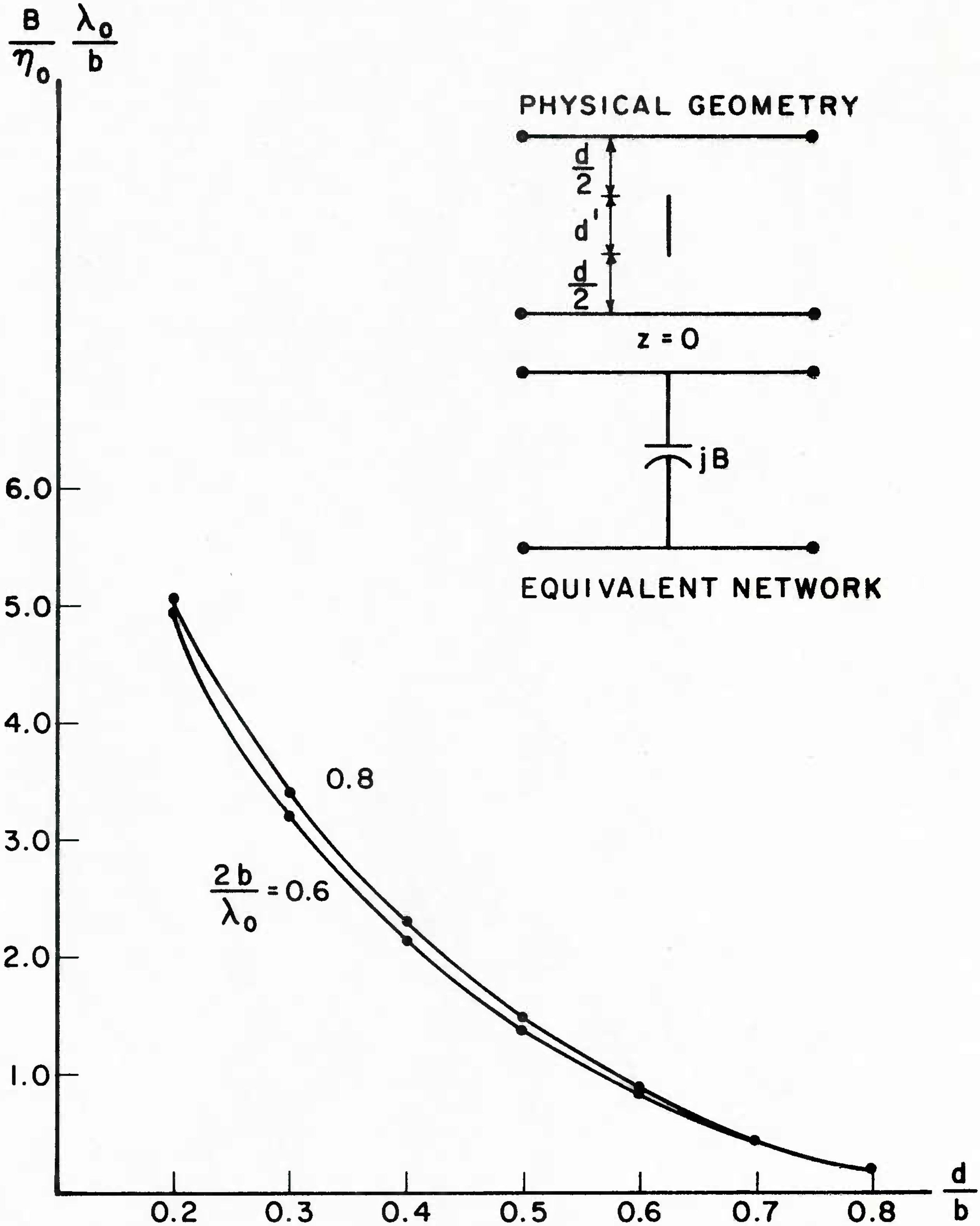

Figure 7. Network susceptance of the symmetrical thin strip. The number of segments used is 20.

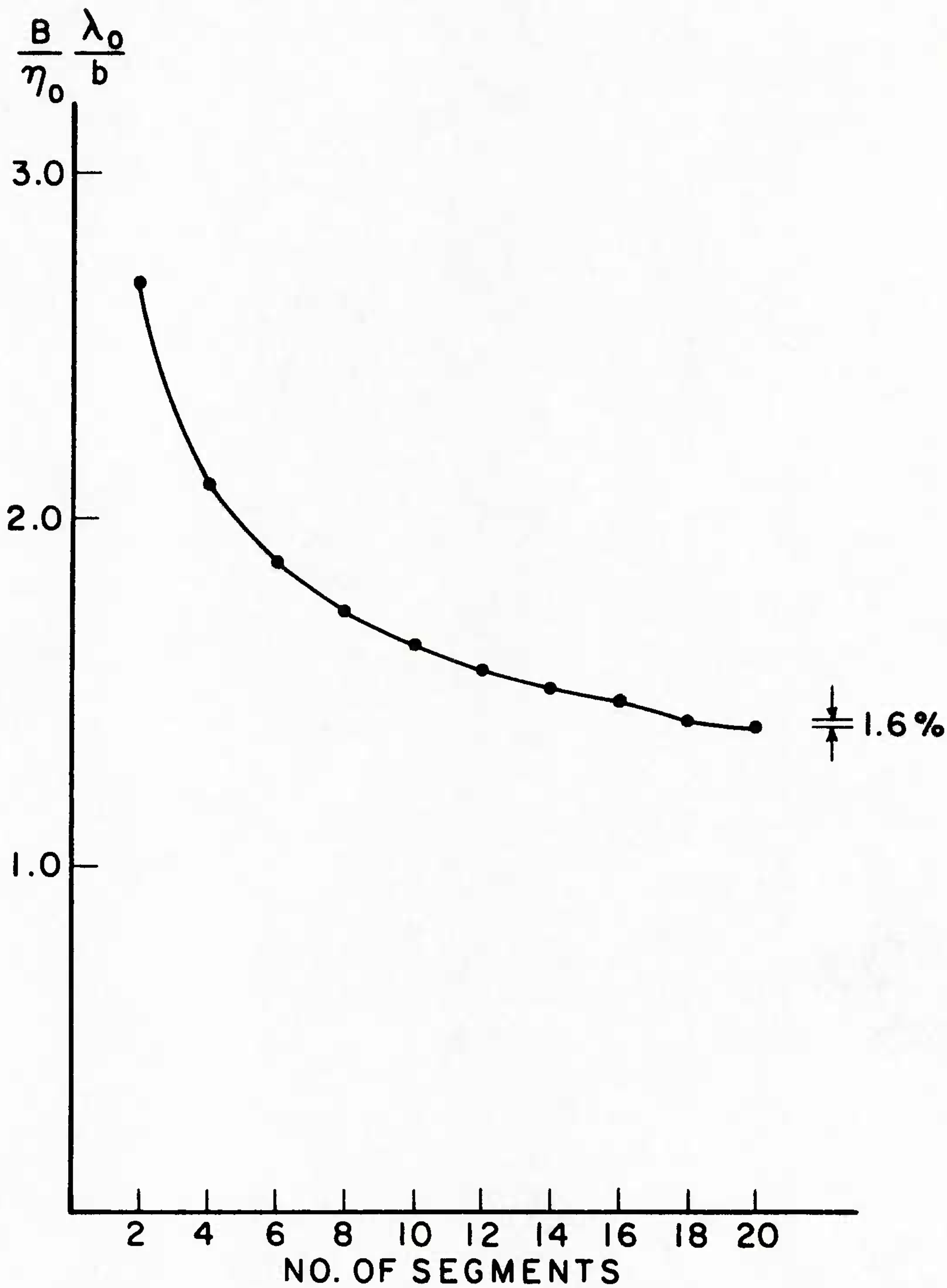

Figure 8. The convergence behavior for the symmetrical thin strip ($\frac{2b}{\lambda_0} = 0.6$, $\frac{d}{b} = 0.5$).

A complete field analysis of the problem is first given. From the analysis, the scattering and impedance matrix representations of the system of posts fully describing its effect on the dominant waveguide mode are obtained. Since the whole structure is both reciprocal and lossless, the scattering and impedance matrices are symmetric and unitary, and symmetric and pure imaginary, respectively. The latter is then realized in the form of a T-network of capacitive elements.

The susceptances of some post configurations are computed in Section 9. In the actual computation, pulse expansions of the currents induced on the posts are used. Although chosen primarily so as to render the procedure economical, the choice is very natural, since pulses are instrumental in the definition of integration [16, Chapter 10]. This choice has proven very successful, nevertheless, as is evident from the performance of the solution.

Circular posts and thin diaphragms and windows were treated early using the Variational Method [1, Chapter 3], [2, Section 8-9]. In [18, Section 6-3], the "Singular Integral Equation Method" was applied to the symmetrical thin window. In contrast to the limited application of these two methods, the present analysis is quite general. It can become the first step in the solution of the system of multiple dielectric posts in the capacitive position in a rectangular waveguide.

Appendix A

Consider the function defined by the series

$$\left.\begin{aligned} G_1 &= \sum_{n=1}^{\infty} \frac{\cos(n\eta)\ \cos(n\eta')\ e^{-n\sigma}}{n} \\ \\ \sigma &= |\xi - \xi'| \ . \end{aligned}\right\} \tag{A.1}$$

Since

$$\cos(n\eta)\ \cos(n\eta') = \frac{1}{2}\ (\cos\ n(\eta-\eta') + \cos\ n(\eta+\eta'))$$

$$= \frac{1}{2}\ \mathrm{Re}\ (e^{-jn(\eta-\eta')} + e^{-jn(\eta+\eta')}) \tag{A.2}$$

(A.1) becomes

$$G_1 = \frac{1}{2}\ \mathrm{Re}(\sum_{n=1}^{\infty} \frac{1}{n}\ e^{-n(\sigma+j(\eta-\eta'))} + \sum_{n=1}^{\infty} \frac{1}{n}\ e^{-n(\sigma+j(\eta+\eta'))}) \ . \tag{A.3}$$

In (A.2) and (A.3), Re(z) denotes the real part of z.

Since

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \ , \qquad z \in C,\ |z| < 1 \tag{A.4}$$

and the series in (A.4) converges uniformly for all z, $|z| \leq |z_o| < 1$, a term by term integration can be carried out [20, Section 5-4], giving

$$\int_0^{z_o} \frac{dz}{1-z} = \sum_{n=0}^{\infty} \int_0^{z_o} z^n\ dz$$

$$-\log\ (1 - z_o) = \sum_{n=0}^{\infty} \frac{z_o^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{z_o^n}{n} \tag{A.5}$$

In (A.5), log denotes the natural logarithm. Putting z_o equal to $e^{-(\sigma+j(\eta-\eta'))}$ and $e^{-(\sigma+j(\eta+\eta'))}$ in (A.5), then using the results in (A.3), G_1 is found to be

$$G_1 = -\frac{1}{2}\,\mathrm{Re}[\log((1 - e^{-(\sigma+j(\eta-\eta'))})(1 - e^{-(\sigma+j(\eta+\eta'))}))]. \qquad (A.6)$$

Finally, since

$$|(1 - e^{-(\sigma+j(\eta-\eta'))})(1 - e^{-(\sigma+j(\eta+\eta'))})|^2$$

$$= e^{-4\sigma}(e^{2\sigma} - 2e^{\sigma}\cos(\eta+\eta')+1)\ (e^{2\sigma} - 2e^{\sigma}\cos(\eta-\eta')+1)$$

$$= 4e^{-2\sigma}\ (\cosh\sigma - \cos(\eta+\eta'))(\cosh\sigma - \cos(\eta-\eta')) \qquad (A.7)$$

G_1 becomes

$$G_1 = \frac{1}{2}\sigma - \frac{1}{4}\log[4(\cosh\sigma - \cos(\eta+\eta'))(\cosh\sigma - \cos(\eta-\eta'))]. \qquad (A.8)$$

Since

$$(1 - e^{-j(\eta+\eta')})(1 - e^{-j(\eta-\eta')})$$

$$= 1 - e^{-j(\eta+\eta')} - e^{-j(\eta-\eta')} + e^{-j2\eta}$$

$$= 2e^{-j\eta}\ (\cos\eta - \cos\eta') \qquad (A.9)$$

G_1, as σ tends to zero, becomes

$$G_1 = -\frac{1}{2}\log\ (2|\cos\eta - \cos\eta'|)\ . \qquad (A.10)$$

The singular part of G_1 can readily be extracted from (A.8). That is, clearly,

$$G_{1s} = -\frac{1}{4}\log(2\ (\cosh\ (\xi-\xi') - \cos\ (\eta-\eta'))) \ . \tag{A.11}$$

Chapter 5

DISCUSSION

Three systems of waveguide discontinuities have been considered in this dissertation.

The first system is that of multiple apertures of arbitrary shape in the transverse plane between two cylindrical waveguides. The second system consists of metallic obstacles in a rectangular waveguide that are uniform along the narrow side of the waveguide, but are otherwise of arbitrary shape and thickness, i.e., a system of inductive posts. Finally, the third system consists of metallic obstacles in a rectangular waveguide that are uniform along the broad side of the waveguide, but are otherwise of arbitrary shape and thickness, i.e., a system of capacitive posts. Common between the first and second systems are the inductive windows and strips in a rectangular waveguide, and between the first and third systems are the capacitive windows and strips in a rectangular waveguide.

The analysis of waveguide discontinuities has primarily been carried out on an elementary scale. That is, solution techniques have been sought and applied for problems with a single discontinuity. Rarely have problems involving three or more discontinuities been considered, and only then if they are all of the same shape. Although some of the elementary problems are of practical importance that warrants considering them, these and others are better worked out as special cases of general systems of discontinuities. This, of course, provided that general and efficient solutions can be found for such systems. It has been the purpose of this dissertation to demonstrate that for the three systems considered.

Other systems of discontinuities can be handled using the methods employed in this dissertation. In fact, the solutions given in Chapters 3 and 4 can become the first step in the analysis of systems of multiple dielectric posts in the inductive and capacitive position in a rectangular waveguide, respectively. Another important system of discontinuities, but one for which major revisions in the analysis of Chapter 3 are to be made, is that of metallic resonant posts in the inductive position in a rectangular waveguide. These systems, as well as others that can be identified, are recomended for future study.

It is believed that advances can be made in the study of the waveguide discontinuities by considering systems of discontinuities rather than individual discontinuities. General and efficients solutions can be developed for these systems as, it is hoped, has been demonstrated in this dissertation.

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